Approximated analytical solutions of SW equations for particle-driven gravity currents propagating in channels of power-law cross-sections

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Introduction/Motivation

“Particle-driven gravity current” is the name of common effect: suspension of dense particles moves in horizontal direction into an ambient fluid because the densities are different.

Volcanic eruptions

Turbidity gravity currents:
Introduction/Motivation: Examples

Parabolic corridor  Valley  Triangle corridor
Schematic view - Typical channels

\[ y = f(z) \]

Cross-section area:

\[ A_c(h) = \int_0^h f(z) \, dz; \]

\[ A'_c(h) = f(h); \]

\[ A_T = \int_0^H f(z) \, dz \]
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Cross-section area:

\[ A_c(h) = \int_0^h (x, t) f(z) \, dz; \quad A'_c(h) = f(h); \quad A_T = \int_0^H f(z) \, dz \]
Schematic view - Side view
System parameters

**Ambient fluid:** density = $\rho_a = \text{const}$;
System parameters

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Interstitial fluid: density $= \rho_i$; viscosity $= \nu$

Particles:

- density $\rho_p$; radius $a_p$;

- Stokesian settling speed $W_s = \frac{2}{9} \frac{\rho_p - \rho_i}{\rho_i} \frac{a_p^2}{\nu} g$

- Volume fraction for concentration of the particles: $\kappa(x, z, t)$. Initially $\kappa_0$;
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- Volume fraction for concentration of the particles: \( \kappa(x, z, t) \). Initially \( \kappa_0 \);

Density of Current: \( \rho_c = (1 - \kappa)\rho_i + \kappa \rho_p \);

Effective Reduced gravity: \( g'_e = (\frac{\rho_c}{\rho_a} - 1)g = \epsilon_p \kappa_0 \frac{\kappa}{\kappa_0} g \), where \( \epsilon_p = \frac{\rho_p - \rho_i}{\rho_i} \)
Assumptions and Analysis

Assume: inviscid, \( Re \gg 1 \), Boussinesq, thin layer, negligible return flow
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\[ \frac{dV}{dt} = 0 \]
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Analysis: The pressures are hydrostatic and

\[ \frac{\partial \rho_c}{\partial x} = \rho_a \varepsilon_p g \left[ \kappa(x, t) \frac{\partial h}{\partial x} + (h - z) \frac{\partial \kappa}{\partial x} \right]. \]
Governing equations and Front condition

* The volume **continuity equation** obtained using **geometric** considerations;
* The **x-momentum equation** is obtained by the usual averaging, plus the \( \partial p_c/\partial x(x,t) \) term
* The "**diffusion**" equation for volume fraction in the suspension

What happens at \( x_N(t) \)?
Governing equations and Front condition

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What happens at \( x_N(t) \)?

\[
u_N = (g_e')^{1/2} h_N^{1/2} Fr(\phi).
\]
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* The volume continuity equation obtained using geometric considerations;
* The $x$-momentum equation is obtained by the usual averaging, plus the $\partial p_c / \partial x(x, t)$ term
* The "diffusion" equation for volume fraction in the suspension

What happens at $x_N(t)$?

$$u_N = (g'_e)^{1/2} h_N^{1/2} Fr(\varphi).$$

$Fr(\varphi)$ is an “off the shelf” formula (Ungarish (2011)):

$$Fr = Fr_U(\varphi) = \left[ \frac{2(1-\varphi)^2}{1+\varphi} (1+Q) \right]^{1/2},$$

where

$$\varphi = \frac{A_c(h)}{A_T}, \text{ and } Q = \frac{\int_0^h zf(z)dz}{h \cdot [A_T - A_c(h)]}. \quad (1)$$
SW equations

Scaling:
- \{z, x, y\} w. \{h_0, x_0, f(h_0)\} of lock;
- u w. \(U = (\varepsilon_p \kappa_0 g h_0)^{1/2}\);
- \(t\) w. \(x_0/U\); scaled volume fraction: \(\phi = \frac{\kappa}{\kappa_0}\) in [0, 1]
- scaled Stokes settling velocity of the particles: \(\beta = \frac{W_s}{U} \frac{x_0}{h_0}\) (\(\beta \ll 1\))

Equations of motion:

\[
\begin{pmatrix}
    h_t \\
    u_t \\
    \phi_t
\end{pmatrix} + \begin{pmatrix}
    u & \frac{h}{\alpha + 1} & 0 \\
    \phi & u & \frac{h}{\alpha + 2} \\
    0 & 0 & u
\end{pmatrix} \begin{pmatrix}
    h_x \\
    u_x \\
    \phi_x
\end{pmatrix} = \begin{pmatrix}
    0 \\
    0 \\
    -\beta \phi \frac{\alpha + 1}{h}
\end{pmatrix}.
\]

(2)

The system is hyperbolic; IC and BC are known
Finite-Difference results for $f(z) = z^\alpha$

Lax-Wendroff 2l method with 200 grid points: $\delta t = 10^{-3}$; $f(z) = z$; $\beta = 2.5 \times 10^{-3}$; $H = 10$

$h$ vs. $t$: $t = 1(1)10(10), 100$
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\[ h \text{ vs. } t : t = 1(1)10(10), 100 \]

**Slumping stage**

* Constant \( u_N \) and \( h_N \).

* \( x_s \) increases with \( \alpha \).

* As \( \alpha \) increases: current faster; slumping stage - longer.
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**Transient stage** Decreasing height of the nose and speed.
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**Transient stage** Decreasing height of the nose and speed.

**Self-Similarity** Long-time profiles with "tail down - nose up" height form.
Finite-Difference results for $f(z) = z^\alpha$

Volume fraction for $f(z) = z$

$\phi$ vs. $t : t = 1(1)10(10), 100$
Comparison with Experiments

Experimental results of Mériaux et al.[2015] (symbols) and SW prediction (line) for $H = 1$, $f(z) = z$. 
Box-Model

**Box-model approximation:** the current has a simple shape of length $x_N(t)$ and uniform height $h_N(t)$; $Fr = const$
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Box-Model Equations for power-law $f(z) = z^\alpha$

Conservation of the Particles and B.C.:

\[
\begin{cases}
\frac{d(V\phi)}{dt} = -\beta \phi x_N h_N^\alpha \\
\frac{dx_N}{dt} = Fr \phi^{1/2} h_N^{1/2}
\end{cases}
\]  

Here

\[
\begin{cases}
V(t) = \int_0^{x_N} A(h)dx = h^\alpha x_N \\
Fr = \text{const;}
\end{cases}
\]  

$V = V_0$ and $\phi$ decays.

\[
x_{\text{max}}^{(T)} \approx (2\alpha + 5)^{\frac{2\alpha+2}{2\alpha+5}} (\alpha + 1)^{-\frac{2\alpha}{2\alpha+5}} \left(\frac{Fr}{\beta}\right)^{\frac{2\alpha+2}{2\alpha+5}} V_0^{\frac{3}{2\alpha+5}}
\]
Similarity solution

- Large times after release, the current "forgets" the initial conditions.
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- \( Fr \approx \sqrt{2} \)
- The solutions are:

\[
    x_N(t) = K t^\gamma, \quad h_N(t, y) = (\dot{x}_N)^2 \mathcal{H}(y), \\
    u_N(t, y) = \dot{x}_N \mathcal{U}(y),
\]

where

\[
    y = \frac{x}{x_N};
\]

And \( K \) and \( \gamma \) are constants.
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where

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y = \frac{x}{x_N};
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And \( K \) and \( \gamma \) are constants.

- The formulation above satisfies the nose condition \( u_N = \dot{x}_N = Fr \cdot h_N^{1/2} \)
Similarity solution for Homogeneous case

\[ x_N = K t^\gamma, \quad h = \gamma^2 K^2 t^{2\gamma - 2} H(y), \quad u = \gamma K t^{\gamma - 1} U(y), \quad \phi = 1, \]
Similarity solution for Homogeneous case

\[ x_N = Kt^\gamma, \quad h = \gamma^2 K^2 t^{2\gamma - 2} H(y), \quad u = \gamma K t^{\gamma - 1} U(y), \quad \phi = 1, \]

where

\[ y = \frac{x}{x_N} = \frac{x}{K t^\gamma}; \quad \gamma = \frac{2 + 2\alpha}{3 + 2\alpha} \]
Similarity solution for Homogeneous case

\[ x_N = K t^\gamma, \quad h = \gamma^2 K^2 t^{2\gamma-2} H(y), \quad u = \gamma K t^{\gamma-1} U(y), \quad \phi = 1, \]

where

\[ y = \frac{x}{x_N} = \frac{x}{K t^\gamma}; \quad \gamma = \frac{2 + 2\alpha}{3 + 2\alpha} \]

\[ U(y) = U_0(y) = y, \]

\[ H(y) = H_0(y) = \frac{1}{Fr^2} - \frac{1}{4(\alpha + 1)} + \frac{1}{4(\alpha + 1)} \cdot y^2, \]

\[ K = \left( \frac{1}{\gamma} \right)^\gamma \left( \int_0^1 \frac{1}{H^{\alpha+1}(y)} \, dy \right)^{1/(2\alpha + 3)} \]
We use following expansions in the regime \( \tau \ll 1 \):

\[
\begin{align*}
\xi_N &= K t^\gamma (1 + \tau X_1 + \tau^2 X_2 + \ldots), \\
u &= \gamma K t^{\gamma-1} (U_0(y) + \tau U_1(y) + \tau^2 U_2(y) + \ldots), \\
h &= \gamma^2 K^2 t^{2\gamma-2} (H_0(y) + \tau H_1(y) + \tau^2 H_2(y) + \ldots), \\
\phi &= 1 + \tau \phi_1(y) + \tau^2 \phi_2(y) + \ldots,
\end{align*}
\]

where

\[
\tau = \beta K^{-2} t^{3-2\gamma} = \beta K^{-2} t^{(5+2\alpha)/(3+2\alpha)}.
\]
Substitution of expansions in equations of motion and b.c:

\[
\left( \frac{1}{Fr^2} - \frac{1}{4(\alpha + 1)} + \frac{y^2}{4(\alpha + 1)} \right) H_1'' + \frac{1}{2} y H_1' - \frac{(2\alpha + 5)(2\alpha + 3)}{2(\alpha + 1)} H_1 = F(y),
\]

where \( F(y) \) is function of \( y \)

\[
F(y) = A \left( H_0'' + \alpha \frac{(H_0')^2}{H_0} \right).
\]

The boundary conditions:

\[
H_1'(0) = 0;
\]

\[
H_1(1) + \frac{(14 + 8\alpha - Fr^2)(\alpha + 1)}{(5 + 2\alpha)(3 + 2\alpha)Fr^2} H_1'(1) = \frac{(3 + 2\alpha)^3}{4(5 + 2\alpha)(1 + \alpha) \left[ 1 + \frac{1}{2} \frac{(14 + 8\alpha - Fr^2)}{(3 + 2\alpha)(5 + 2\alpha)} \right]}
\]
Upon the transformation of the independent variable,

\[ \zeta = iy \left( \frac{4(\alpha + 1)}{Fr^2} - 1 \right)^{-1/2}, \]  

(8)

(6) is reduced to

\[
(1 - \zeta^2)H_1'' - 2(\alpha + 1)\zeta H_1' + 2(2\alpha + 5)(2\alpha + 3)H_1 = -4(\alpha + 1)F(\zeta),
\]  

(9)

where \(F(\zeta)\) is obtained from \(F(y)\) by substitution of (8) into (7). This is a standard ultraspherical or Gegenbauer second-order differential equation. Its general solution is expressed by:

\[
H_1(\zeta) = (\zeta^2 - 1)^{-\alpha/2} \left[ C_1 P_{3\alpha+5}^{-\alpha}(\zeta) + C_2 Q_{3\alpha+5}^{-\alpha}(\zeta) \right] + H_1^p(\zeta),
\]  

(10)

where \(P_{3\alpha+5}^{-\alpha}(\zeta)\) and \(Q_{3\alpha+5}^{-\alpha}(\zeta)\) are associated Legendre functions of the first and second kinds. And \(H_1^p(\zeta)\) is the particular solution of (9).
Asymptotic similarity solutions $\alpha = 0$

\[ H_1(\zeta) = -0.023Q_5(\zeta) - \frac{9}{200} \]
Asymptotic similarity solutions $\alpha = 1$

\[ H_1(\zeta) = -0.403(1 - \zeta^2)^{-1/2} Q_{8}^{-1}(\zeta) + \frac{125}{1512} \cdot \frac{1}{1 - \zeta^2} - \frac{25}{196} \]

(12)
Asymptotic similarity solutions $\alpha = 2$

$$H_1(\zeta) = -6.489 \cdot \frac{1}{1 - \zeta^2} Q_{11}^{-2}(\zeta) - 0.041 + 0.17\zeta^2$$ (13)
Asymptotic similarity solutions $H_1(y)$ and $U_1(y)$

\[ U_1(y) = -\frac{\beta}{2} \left[ H_1'(y) - A \frac{H_0'}{H_0} \right], \quad (14) \]

where

\[ A = \frac{(\alpha + 1)^2}{\alpha + 2} \cdot \frac{1}{\beta^2(3 - 2\beta)} \quad (15) \]
Asymptotic similarity solutions $\phi_1(y)$

$$\phi_1 = -\frac{(3 + 2\alpha)^3}{4(5 + 2\alpha)(1 + \alpha)} / H_0(y).$$

(16)
Bi-Disperse currents
System parameters

Ambient fluid: density = $\rho_a = \text{const}$;

Interstitial fluid: density = $\rho_i$; viscosity = $\nu$

Particles:

- 2 types of particles
- Type 1: density $\rho_p^{(1)}$, radius $a_p^{(1)}$, concentration $\kappa^{(1)}(x, z, t)$ (init $\kappa_0^{(1)}$);
- Type 2: density $\rho_p^{(2)}$, radius $a_p^{(2)}$, concentration $\kappa^{(2)}(x, z, t)$ (init $\kappa_0^{(2)}$);
- Density ratio parameters: $\varepsilon_p^{(j)} = \frac{\rho_p^{(j)}}{\rho_a} - 1$ (j = 1, 2).
- Stokesian settling speed $W_s^{(j)} = \frac{2 \varepsilon_p^{(j)} (a_p^{(j)})^2}{\nu} g$, (j = 1, 2).

Density of Current: $\rho_c = \rho_a (1 + \kappa^{(1)} \varepsilon_p^{(1)} + \kappa^{(2)} \varepsilon_p^{(2)})$
Shallow-water equations of motion

\[
\begin{pmatrix}
    h \\
    u \\
    \phi^{(1)} \\
    \phi^{(2)}
\end{pmatrix}_t + \begin{pmatrix}
    u & \frac{h}{\alpha+1} & 0 & 0 \\
    \phi^{(1)} + \phi^{(2)} & u & \frac{h}{\alpha+2} & \frac{h}{\alpha+2} \\
    0 & 0 & u & 0 \\
    0 & 0 & 0 & u
\end{pmatrix} \begin{pmatrix}
    h \\
    u \\
    \phi^{(1)} \\
    \phi^{(2)}
\end{pmatrix}_x = \begin{pmatrix}
    0 \\
    0 \\
    -\beta^{(1)} \phi^{(1)} & \alpha+1 \\
    -\beta^{(2)} \phi^{(2)} & \alpha+1
\end{pmatrix}.
Finite-Difference results for $f(z) = z^1$

We use Lax-Wendroff 2L method with 200 grid points; $\delta t = 10^{-3}$; $f(z) = z$; $H = 10$, $\beta_1 = 2.5 \times 10^{-3}$, $\beta_2 = 7\beta_1$, $\phi_1(0) = 0.8$.

$h$ vs. $t$: $t = 2(2)10, 20(10), 70$

$\phi_1$ vs. $t$: $t = 2(2)10, 20(10), 70$
Finite-Difference results for $f(z) = z^2$

$h$ vs. $t$: $t = 2(2)10, 20(10), 70$

$\phi_1$ vs. $t$: $t = 2(2)10, 20(10), 70$
Comparison with Experiments

Experimental results of Mériaux et al.[2016] (symbols) and SW prediction (line) for triangle cross-section. $H = 1$, $f(z) = z$. Also shown the SW solution for average values of $\bar{\beta} = \sum_{j=1}^{2} \beta^{(j)} \phi^{(j)}(0)$. (dashed line).
Box-Model Runout for power-law $f(z) = z^\alpha$

$V = V_0$ and $\phi_1$ decays.

\[ x_{\text{max}} \approx \left( \frac{Fr(2\alpha + 5)}{2\beta_1} \right)^{\frac{2\alpha+2}{2\alpha+5}} V_0^{\frac{3}{2\alpha+5}} (\alpha + 1)^{-\frac{4\alpha+1}{2\alpha+5}} \int_0 V_0 \phi^{(1)}(0) \left[ \sum_{j=1}^2 a_j \cdot \frac{y^{\beta^{(j)}}/\beta^{(1)}}{y} \right]^{1/2} dy \]  

where

\[ a_j = \frac{\phi^{(j)}(0)}{(\phi^{(1)}(0) \cdot V)^{\beta^{(j)}/\beta^{(1)}}}, ((j = 1, 2)) \]  

(17)
Box-Model Prediction of Runout distance for \( f(z) = z^\alpha \)

\[ \alpha = 0, 0.5, 1, 2; \quad H = 10, \beta_1 = 2.5 \times 10^{-3}, \beta_2 = 7\beta_1. \]

Runout decreases with increasing of coarse particles concentration.
Box-Model Prediction of Runout distance for $f(z) = z^\alpha$

$\alpha = 0, 0.5, 1, 2; H = 10, \beta_1 = 2.5 \times 10^{-3}, \beta_2 = 7\beta_1.$

Runout decreases with increasing of coarse particles concentration

Runout normalized by mono $\bar{\beta}$: Degree of polydispersion increases the runout.
Summary

- **Mono-disperse currents**: Analytical expression for runout distance and Analytical Similarity solution
- **Poly-disperse currents**: Analytical expression for runout distance
thank you