Quasilinear elliptic equations with gradient dependence

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1joint work with D. Averna and D. Motreanu
Nonlinear Dirichlet problem driven the $(p, q)$-Laplacian operator

\[
\begin{cases}
-\Delta_p u - \mu \Delta_q u = f(x, u, \nabla u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases} \quad (P_\mu)
\]

- $\Omega \subset \mathbb{R}^N$ is a nonempty bounded open set with the boundary $\partial \Omega$
- $\mu$ positive real parameter
- $1 < q < p,$
- $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$
- $\Delta_q u = \text{div}(|\nabla u|^{q-2}\nabla u),$
- $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$, is a Carathéodory function
  - $f(\cdot, s, \xi)$ is measurable for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$
  - $f(x, \cdot, \cdot)$ is continuous for a.e. $x \in \Omega.$
The case in which the nonlinear term does not depend on the gradient $\nabla u$

\[
\begin{cases}
-\Delta_p u - \mu \Delta_q u = f(x, u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]

has been studied by using variational methods


If $\mu = 0$

\[ \begin{aligned}
-\Delta_p u &= f(x, u, \nabla u) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial\Omega
\end{aligned} \]  \quad (P_0)


when $\mu = 1$

\[ \begin{aligned}
-\Delta_p u - \Delta_q u &= f(x, u, \nabla u) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial\Omega
\end{aligned} \]

Under suitable assumption on $f$ we want to prove

- Existence of solutions by using the theory of pseudomonotone operator
- Asymptotic properties as $\mu \to 0^+$ and $\mu \to +\infty$
- Uniqueness of solutions
- Location of solutions by using the method of sub-solution and super-solution for quasilinear elliptic equations combined with comparison arguments.

We refer to following books for details related to pseudomonotone operator and to the method of subsolution-supersolution.


Definition

Let $A : X \to X^*$. We say that $A$ has $S_+$-property iff every sequence $\{u_n\} \subset X$ such that $u_n \rightharpoonup u$ in $X$ and \( \limsup_{n \to +\infty} \langle Au_n, u_n - u \rangle \leq 0 \) implies that $u_n \to u$ in $X$.

Definition

$A : X \to X^*$ is called **pseudomonotone** if $u_n \rightharpoonup u$ and \( \limsup_{n \to +\infty} \langle Au_n, u_n - u \rangle \leq 0 \) imply that $Au_n \rightharpoonup Au$ and $\langle Au_n, u_n \rangle \to \langle Au, u \rangle$.

Consider the negative $p$-Laplacian

$$-\Delta_p : W^{1,p}_0(\Omega) \to W^{-1,p'}_0(\Omega)$$

is continuous, bounded, pseudomonotone and has the $S_+$-property.

the first eigenvalue of $p$-Laplacian operator admits the following variational characterization

$$\lambda_{1p} = \inf_{u \in W^{1,p}_0(\Omega), u \neq 0} \frac{\|\nabla u\|_{L^p(\Omega)}^p}{\|u\|_{L^p(\Omega)}^p}$$
The nonlinearity $f$ satisfies the following conditions

(H1) There exist constants $a_1 \geq 0$, $a_2 \geq 0$, $\alpha \in [0, p^* - 1]$, $\beta \in [0, \frac{p}{(p^*)'}]$, and a function $\sigma \in L^{\gamma'}(\Omega)$, with $\gamma \in [1, p^*]$, such that

$$|f(x, s, \xi)| \leq \sigma(x) + a_1|s|^{\alpha} + a_2|\xi|^{\beta} \text{ a.e. } x \in \Omega, \ \forall (s, \xi) \in \mathbb{R} \times \mathbb{R}^N;$$

(H2) there exist constants $d_1 \geq 0$, $d_2 \geq 0$ with $\lambda^{-1}_{1,p}d_1 + d_2 < 1$, and a function $\omega \in L^1(\Omega)$ such that

$$f(x, s, \xi)s \leq \omega(x) + d_1|s|^p + d_2|\xi|^p \text{ a.e. } x \in \Omega, \ \forall (s, \xi) \in \mathbb{R} \times \mathbb{R}^N.$$

$$p' = \frac{p}{p - 1} \quad p^* = \begin{cases} \frac{pN}{N - p} & p < N \\ +\infty & p \geq N \end{cases}$$
The functional space associated to problem is the Sobolev space $W^{1,p}_0(\Omega)$ with the norm

$$||u|| := \left(\int_\Omega |\nabla u|^p \, dx \right)^{\frac{1}{p}}$$

for all $u \in W^{1,p}_0(\Omega)$.

A (weak) solution of problem ($P_\mu$) for $\mu \geq 0$ is any $u \in W^{1,p}_0(\Omega)$ such that

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v \, dx + \mu \int_\Omega |\nabla u|^{q-2} \nabla u \nabla v \, dx - \int_\Omega f(x, u, \nabla u) v \, dx = 0$$

for all $v \in W^{1,p}_0(\Omega)$.
The Nemytskii operator associated to $f$

$$N : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$$

defined by

$$N(u) = f(x, u, \nabla u)$$

is well defined, continuous and bounded.

We consider the operator

$$A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$$

$$A(u) = -\Delta_p u - \mu \Delta_q u - N(u), \quad (1)$$

Then

$u \in W_0^{1,p}(\Omega)$ is a weak solution of problem $(P_\mu) \iff A(u) = 0$

**Theorem**

**Main theorem on pseudomonotone operator** Let $X$ be a real reflexive Banach space, $A : X \to X^*$ be a pseudomonotone, bounded and coercive operator. Then there is a solution of the equation $Ax = 0$. 

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Existence of solutions

Theorem

Assume that conditions (H1) and (H2) hold. Then problem \((P_\mu)\), with \(\mu \geq 0\), admits at least one weak solution \(u_\mu \in W_0^{1,p}(\Omega)\).
Proof.

\[ A : W_0^{1,p}(\Omega) \rightarrow W^{-1,p'}(\Omega) \]

\[ A(u) = -\Delta_p u - \mu \Delta_q u - N(u), \]

1. \( A : W_0^{1,p}(\Omega) \rightarrow W^{-1,p'}(\Omega) \) is bounded, which means that it maps bounded sets onto bounded sets.

2. for every sequence \( \{u_n\} \subset W_0^{1,p}(\Omega) \) such that \( u_n \rightharpoonup u \) in \( W_0^{1,p}(\Omega) \), by using that the operator \( -\Delta_p - \mu \Delta_q \) on the space \( W_0^{1,p}(\Omega) \) has the \( S_+ \)-property, we have \( u_n \rightarrow u \).

3. \( A \) is pseudomonotone

4. \( A \) is coercive

\[ \lim_{\|u\| \rightarrow \infty} \frac{\langle Au, u \rangle}{\|u\|} = +\infty. \]

Since \( A : W_0^{1,p}(\Omega) \rightarrow W^{-1,p'}(\Omega) \) is pseudomonotone, bounded and coercive, we can apply the main theorem on pseudomonotone operators. Therefore there is at least one element \( u_\mu \in W_0^{1,p}(\Omega) \) such that \( Au_\mu = 0 \), so \( u_\mu \) is a weak solution of problem \( (P_\mu) \), which completes the proof.
Problem $(P_\mu)$ possesses a solution $u_\mu \in W^{1,p}_0(\Omega)$ for every $\mu > 0$. We establish the following a priori estimate.

**Lemma**

Assume that conditions (H1) and (H2) hold. Then there exists a constant $b > 0$ independent of $\mu > 0$ such that

$$\|\nabla u_\mu\|_{L^p(\Omega)} \leq b, \quad \forall \mu > 0. \quad (2)$$

where

$$b = \left( \frac{\|\omega\|_{L^1(\Omega)}}{1 - d_1 \lambda_{1p}^{-1} - d_2} \right)^{\frac{1}{p}}$$
Asymptotic properties as \( \mu \to 0 \)

We consider the limit points \((u_\mu)\) as \( \mu \to 0 \) in problem \((P_\mu)\).

**Theorem**

For any sequence \( \mu_n \to 0^+ \), there exists a relabeled subsequence of solutions \((u_{\mu_n})\) of the corresponding problems \((P_{\mu_n})\) such that \( u_{\mu_n} \to u \) in \( W^{1,p}_0(\Omega) \), with \( u \in W^{1,p}_0(\Omega) \) weak solution of problem \((P_0)\)

\[
\begin{aligned}
-\Delta_p u &= f(x, u, \nabla u) \quad \text{in } \Omega, \\
 u &= 0 \quad \text{on } \partial\Omega
\end{aligned}
\]
Proof.
Since $u_n$ is a weak solution of problem $(P_{\mu_n})$, from Lemma
\(\{u_n\}\) is bounded in $W_0^{1,p}(\Omega)$. Then there exists a subsequence $\{u_{k_n}\}$ such that $u_{k_n} \rightharpoonup u$. From $(P_{\mu_{k_n}})$ we obtain
\[
\lim_{k_n \to +\infty} \langle -\Delta_p u_{k_n}, u_{k_n} - u \rangle = 0
\]
Since $-\Delta_p$ satisfies the $S_+$-property, we have $u_{k_n} \to u$.
Letting $k_n \to +\infty$ in $(P_{\mu_{k_n}})$ it is easy to see that $u$ is a weak solution of problem $(P_0)$.
Asymptotic properties as $\mu \to +\infty$

We consider $\mu \to +\infty$ and the problem

\[
\begin{aligned}
-\frac{1}{\mu} \Delta_p u - \Delta_q u &= \frac{1}{\mu} f(x, u, \nabla u) \quad \text{in } \Omega, \\
 u &= 0 \quad \text{on } \partial \Omega
\end{aligned}
\]

($P_{\frac{1}{\mu}}$)

observe that the solutions of problem ($P_\mu$) are solutions of problem ($P_{\frac{1}{\mu}}$).

**Theorem**

*For any sequence $\mu_n \to +\infty$, the sequence of solutions $(u_{\mu_n})$ of the corresponding problems ($P_{\mu_n}$) satisfies $u_{\mu_n} \to 0$ in $W^{1,q}_0(\Omega)$.***
Proof.
Proceeding as in the proof of previous Theorem, we set \( u_n := u_{\mu_n} \) and apply Lemma to derive that the sequence \((u_n)\) is bounded in \( W^{1,p}_0(\Omega) \), so up to a relabeled subsequence we have \( u_n \rightharpoonup u \) in \( W^{1,p}_0(\Omega) \) for some \( u \in W^{1,p}_0(\Omega) \).
We note that \( u_n \) satisfies

\[
\begin{cases}
- \frac{1}{\mu_n} \Delta_p u_n - \Delta_q u_n = \frac{1}{\mu_n} f(x, u_n, \nabla u_n) & \text{in } \Omega, \\
 u_n = 0 & \text{on } \partial \Omega.
\end{cases}
\]  

(3)

If we act with \( u_n - u \) in (3), we find that

\[
\lim_{n \to +\infty} \langle -\Delta_q u_n, u_n - u \rangle = 0.
\]

The \( S_+ \)-property of the operator \(-\Delta_q : W^{1,q}_0(\Omega) \to W^{-1,q'}(\Omega)\) guarantees that \( u_n \to u \) in \( W^{1,q}_0(\Omega) \). Letting \( n \to \infty \) in (3) entails \( \Delta_q u = 0 \), so \( u = 0 \).
We illustrate this topic by presenting a uniqueness result in the case where $p = 2$ or $q = 2$. Our assumption is as follows:

(U1) there exists a constant $b_1 \geq 0$ such that

$$ (f(x, s, \xi) - f(x, t, \xi))(s - t) \leq b_1|s - t|^2 \text{ a.e. } x \in \Omega, \forall \xi \in \mathbb{R}^N, \forall s, t \in \mathbb{R}; $$

(U2) there exist a function $\tau \in L^\delta(\Omega)$, with some $\delta \in [1, p^*[$, and a constant $b_2 \geq 0$ such that the function $f(x, s, \cdot) - \tau(x)$ is linear and

$$ |f(x, s, \xi) - \tau(x)| \leq b_2|\xi| \text{ a.e. } x \in \Omega, \forall (s, \xi) \in \mathbb{R} \times \mathbb{R}^N. $$
Theorem

Assume that conditions (H1), (H2), (U1) and (U2) hold.

(i) If \( p = 2 > q > 1 \) and

\[
b_1 \lambda_{1,2}^{-1} + b_2 \lambda_{1,2}^{-\frac{1}{2}} < 1
\]

then the solution of problem \((P_\mu)\) is unique for every \( \mu > 0 \).

(ii) If \( p > q = 2 \), then the solution of problem \((P_\mu)\) is unique for every

\[
\mu > b_1 \lambda_{1,2}^{-1} + b_2 \lambda_{1,2}^{-\frac{1}{2}}
\].
Proof.

Suppose that $v_\mu \in W^{1,p}_0(\Omega)$ is a second solution of $(P_\mu)$. Acting with $u_\mu - v_\mu$ on the equation in $(P_\mu)$ gives

\[
\langle -\Delta_p u_\mu + \Delta_p v_\mu, u_\mu - v_\mu \rangle + \mu \langle -\Delta_q u_\mu + \Delta_q v_\mu, u_\mu - v_\mu \rangle = \int_\Omega (f(x, u_\mu, \nabla u_\mu) - f(x, v_\mu, \nabla u_\mu))(u_\mu - v_\mu) \, dx
\]

\[
+ \int_\Omega (f(x, v_\mu, \nabla u_\mu) - f(x, v_\mu, \nabla v_\mu))(u_\mu - v_\mu) \, dx.
\]

(i) For $p = 2$, hypotheses (U1) and (U2), in conjunction with (4) and the monotonicity of $-\Delta_q$, imply

\[
\| \nabla (u_\mu - v_\mu) \|_{L^2(\Omega)}^2 \leq b_1 \| u_\mu - v_\mu \|_{L^2(\Omega)}^2 + \int_\Omega (f(x, v_\mu, \nabla (\frac{1}{2}(u_\mu - v_\mu)^2)) \, dx
\]

\[
\leq (b_1 \lambda_{1,2}^{-1} + \frac{b_2}{2}) \| \nabla (u_\mu - v_\mu) \|_{L^2(\Omega)}^2.
\]

Using that $b_1 \lambda_{1,2}^{-1} + b_2 \lambda_{1,2}^{-\frac{1}{2}} < 1$, the equality $u_\mu = v_\mu$ follows.

(ii) For $p > q = 2$, arguing as in the case of part (i), we find the estimate

\[
\mu \| \nabla (u_\mu - v_\mu) \|_{L^2(\Omega)}^2 \leq (b_1 \lambda_{1,2}^{-1} + b_2 \lambda_{1,2}^{-\frac{1}{2}}) \| \nabla (u_\mu - v_\mu) \|_{L^2(\Omega)}^2.
\]

The conclusion that $u_\mu = v_\mu$ ensues provided that $b_1 \lambda_{1,2}^{-1} + \frac{b_2}{2} < \mu$. \qed
Our main goal is to obtain a solution \( u \in W^{1,p}_0(\Omega) \) of problem \((P_\mu)\) with the location property \( u \leq \bar{u} \leq \underline{u} \) a.e. in \( \Omega \), where \( \underline{u} \) and \( \bar{u} \) are subsolution and supersolution of problem \((P_\mu)\).

\( \bar{u} \in W^{1,p}(\Omega) \) is a \textit{supersolution} for problem \((P_\mu)\) if \( \bar{u} \geq 0 \) on \( \partial \Omega \) and

\[
\int_{\Omega} \left( |\nabla \bar{u}|^{p-2} \nabla \bar{u} + \mu |\nabla \bar{u}|^{q-2} \nabla \bar{u} \right) \nabla v \, dx \geq \int_{\Omega} f(x, \bar{u}, \nabla \bar{u}) v \, dx
\]

for all \( v \in W^{1,p}_0(\Omega), \, v \geq 0 \) a.e. in \( \Omega \).

\( \underline{u} \in W^{1,p}(\Omega) \) is a \textit{subsolution} for problem \((P_\mu)\) if \( \underline{u} \leq 0 \) on \( \partial \Omega \) and

\[
\int_{\Omega} \left( |\nabla \underline{u}|^{p-2} \nabla \underline{u} + \mu |\nabla \underline{u}|^{q-2} \nabla \underline{u} \right) \nabla v \, dx \leq \int_{\Omega} f(x, \underline{u}, \nabla \underline{u}) v \, dx
\]

for all \( v \in W^{1,p}_0(\Omega), \, v \geq 0 \) a.e. in \( \Omega \).
Given a subsolution \( u \in W^{1,p}(\Omega) \) and a supersolution \( \bar{u} \in W^{1,p}(\Omega) \) for problem \((P_\mu)\) with \( u \leq \bar{u} \) a.e. in \( \Omega \), we assume that \( f : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R} \) satisfies the growth condition:

\[ (H) \quad \text{There exist a function } \sigma \in L^{\gamma'}(\Omega) \text{ for } \gamma' = \frac{\gamma}{\gamma - 1} \text{ with } \gamma \in (1, p^*) \text{ and constants } a > 0 \text{ and } \beta \in [0, \frac{P}{\gamma - 1}) \text{ such that} \]

\[ |f(x, s, \xi)| \leq \sigma(x) + a|\xi|^\beta \quad \text{for a.e. } x \in \Omega, \text{ all } s \in [u(x), \bar{u}(x)], \xi \in \mathbb{R}^N. \]
Theorem

Let \( u \) and \( \bar{u} \) be a subsolution and a supersolution of problem \((P_\mu)\), respectively, with \( u \leq \bar{u} \) a.e. in \( \Omega \) such that hypothesis \((H)\) is fulfilled. Then problem \((P_\mu)\) possesses a solution \( u \in W^{1,p}_0(\Omega) \) satisfying the location property \( u \leq u \leq \bar{u} \) a.e. in \( \Omega \).
Proof.

Consider auxiliary truncated problem depending on a positive parameter $\lambda$ (for any fixed $\mu \geq 0$)

$$
(T_{\lambda, \mu}) - \Delta_p u - \mu \Delta_q u + \lambda B(u) = N(Tu).
$$

where $T$ is the truncation operator $T : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ defined by

$$
Tu(x) = \begin{cases} 
\bar{u}(x) & \text{if } u(x) > \bar{u}(x) \\
\underline{u}(x) & \text{if } \overline{u}(x) \leq u(x) \leq \underline{u}(x) \\
u(x) & \text{if } u(x) < \underline{u}(x),
\end{cases}
$$

which is known to be continuous and bounded.

$\pi$ is the cut-off function $\pi : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$
\pi(x, s) = \begin{cases} 
(s - \overline{u}(x))^{\frac{\beta}{p-\beta}} & \text{if } s > \overline{u}(x) \\
0 & \text{if } \underline{u}(x) \leq s \leq \overline{u}(x) \\
-(u(x) - s)^{\frac{\beta}{p-\beta}} & \text{if } s < \underline{u}(x).
\end{cases}
$$

$B : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is the Nemytskij operator given by $B(u) = \pi(\cdot, u(\cdot))$.
\textbullet{} $N : [u, \bar{u}] \rightarrow W^{-1,p'}(\Omega)$ is the Nemytskij operator determined by the function $f$ in $(P_\mu)$, that is

$$N(u)(x) = f(x, u(x), \nabla u(x)),$$

\textbullet{} for $\lambda > 0$ sufficiently large, there is a solution $u \in W^{1,p}_0(\Omega)$ of problem $(T_{\mu, \lambda})$.

\textbullet{} by using comparison arguments we prove that every solution $u \in W^{1,p}_0(\Omega)$ of problem $(T_{\mu, \lambda})$ satisfies $u \leq u \leq \bar{u}$ a.e. in $\Omega$.

\textbullet{} the solution $u$ of the auxiliary truncated problem $(T_{\lambda, \mu})$ satisfies $Tu = u$ and $B(u) = 0$, so it is a solution of the original problem $(P_\mu)$.
We want to show you a result on the existence of positive solutions to problem \((P_\mu)\).

The idea is to construct a subsolution \( u \in W^{1,p}(\Omega) \) and a supersolution \( \bar{u} \in W^{1,p}(\Omega) \) with \( 0 < u \leq \bar{u} \) a.e. in \( \Omega \) for which previous Theorem can be applied.
we suppose the following assumptions on $f$

\((H3)\) There exist constants $a_0 > 0$, $b > 0$, $\delta > 0$ and $r > 0$, with $r < p - 1$ if $\mu = 0$ and $r < q - 1$ if $\mu > 0$, such that

$$
\left( \frac{a_0}{b} \right)^{\frac{1}{p-r-1}} < \delta
$$

(5)

and

$$
f(x, s, \xi) \geq a_0 s^r - b s^{p-1} \text{ for a.e. } x \in \Omega, \text{ all } 0 < s < \delta, \xi \in \mathbb{R}^N.
$$

(6)

\((H4)\) There exists a constant $s_0 > \delta$, with $\delta > 0$ in \((H3)\), such that

$$
f(x, s_0, 0) \leq 0 \text{ for a.e. } x \in \Omega.
$$

(7)
Our result on the existence of positive solutions for problem \((P_\mu)\) is as follows.

**Theorem**

Assume \((H3), (H4)\) and that

\[
|f(x, s, \xi)| \leq \sigma(x) + a|\xi|^\beta \quad \text{for a.e. } x \in \Omega, \text{ all } s \in [0, s_0], \xi \in \mathbb{R}^N,
\]

with a function \(\sigma \in L^{\gamma'}(\Omega)\) for \(\gamma \in [1, p^*)\) and constants \(a > 0, \beta \in [0, \frac{p}{(p^*)'})\), and \(s_0\) in \((H4)\). Then, for every \(\mu \geq 0\), problem \((P_\mu)\) possesses a positive smooth solution \(u\) satisfying the a priori estimate \(u(x) \leq s_0\) for all \(x \in \Omega\).
Proof.

1. Consider the following auxiliary problem

\[
\begin{aligned}
-\Delta_p u - \mu \Delta_q u + b|u|^{p-2}u &= a_0(u^+)^r & \text{in } \Omega, \\
u &= 0 & \text{on } \partial \Omega.
\end{aligned}
\]  

(8)

2. We prove that there exists a solution \( u \in C^1_0(\overline{\Omega}) \) of problem such that \( u > 0 \) in \( \Omega \).

3. We claim that \( u \) is a subsolution for problem \((P_\mu)\).

4. Hypothesis \((H_4)\) guarantees that \( \bar{u} = s_0 \) is a supersolution of problem \((P_\mu)\).

5. We have \( u < \bar{u} \) in \( \Omega \).

6. The hypothesis \((H)\) is verified by constructed pair \((u, \bar{u})\) of subsolution-supersolution for problem \((P_\mu)\). Therefore previous theorem ensuring the existence of a solution \( u \in W^{1,p}_0(\Omega) \) for the problem \((P_\mu)\), which satisfies the enclosure property \( u \leq u \leq \bar{u} \) a.e. in \( \Omega \).

7. Taking into account that \( u > 0 \), we conclude that the solution \( u \) is positive.

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