Modified midpoint method for delay differential equations - stability analysis

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We consider

\[ y'(t) = ay(t) + by(t - \tau), \quad t \geq 0, \quad \text{(DDE)} \]

where \( a, b, \tau \in \mathbb{R}, \tau > 0. \)

Initial condition

\[ y(t) = \phi(t), \quad t \in [-\tau, 0]. \quad \text{(IC)} \]
Notion of asymptotic stability of DDE

\[ y'(t) = ay(t) + by(t - \tau), \quad t \geq 0, \quad \text{(DDE)} \]

Delay differential equation (DDE) is said to be asymptotically stable (AS) if all its solutions satisfy

\[ \lim_{t \to \infty} y(t) = 0. \]
\[ y'(t) = ay(t) + by(t - \tau), \quad t \geq 0, \quad \text{(DDE)} \]

**Theorem**

*Equation (DDE) is asymptotically stable if and only if*

\[ a \leq b < -a \quad \text{for all} \quad \tau > 0 \quad \text{(1)} \]

*and in addition to the previous*

\[ |a| + b < 0 \quad \text{for} \quad \tau < \frac{\arccos(-a/b)}{(b^2 - a^2)^{1/2}}. \]


\[ y'(t) = ay(t) + by(t - \tau), \quad t \geq 0, \] (DDE)

**Figure:** \( a \leq b < -a \) for all \( \tau > 0; \) \( |a| + b < 0 \) for \( \tau < \frac{\arccos(-a/b)}{(b^2-a^2)^{1/2}}; \) \( [\tau = 1] \)
Discrete case

\[ Y(n + 2) + \alpha Y(n) + \beta Y(n - \ell) = 0, \quad n = 0, 1, 2, \ldots , \quad \text{(DC)} \]

where \( \alpha, \beta \in \mathbb{R} \) and \( \ell \in \mathbb{N} \).

Difference equation (DC) is said to be \textit{asymptotically stable (AS)} if all its solutions satisfy

\[ \lim_{n \to \infty} Y(n) = 0. \]
Preliminary result - three-term difference equation

\[ Y(n + 2) + \alpha Y(n) + \beta Y(n - \ell) = 0, \quad n = 0, 1, 2, \ldots \]  
(DC)

**Theorem**

Let \( \alpha, \beta \) be arbitrary reals such that \( \alpha \beta \neq 0 \).

(i) Let \( \ell \) be even and \( \beta(-\alpha)^{\ell/2 + 1} < 0 \). Then (DC) is asymptotically stable if and only if

\[ |\alpha| + |\beta| < 1. \]  
(2)

(ii) Let \( \ell \) be even and \( \beta(-\alpha)^{\ell/2 + 1} > 0 \). Then (DC) is asymptotically stable if and only if either

\[ |\alpha| + |\beta| \leq 1, \]  
(3)

or

\[ ||\alpha| - |\beta|| < 1 < |\alpha| + |\beta|, \quad \ell < 2 \arccos \frac{\alpha^2 + \beta^2 - 1}{2|\alpha\beta|} / \arccos \frac{\alpha^2 - \beta^2 + 1}{2|\alpha|^2} \]  
(4)

holds.

Preliminary result - three-term difference equation

\[ Y(n + 2) + \alpha Y(n) + \beta Y(n - \ell) = 0, \quad n = 0, 1, 2, \ldots, \ (DC) \]

where \( \alpha, \beta \in \mathbb{R} \) and \( \ell \in \mathbb{N} \).

**Theorem**

(i) Let \( \ell \) be odd and \( \alpha < 0 \). Then (DC) is asymptotically stable if and only if

\[ |\alpha| + |\beta| < 1. \]

holds.

(ii) Let \( \ell \) be odd and \( \alpha > 0 \). Then (DC) is asymptotically stable if and only if either

\[ |\alpha| + |\beta| \leq 1, \]

or

\[ \beta^2 < 1 - \alpha < |\beta|, \quad \ell < 2 \arcsin \frac{1 - \alpha^2 - \beta^2}{2|\alpha\beta|} / \arccos \frac{\alpha^2 - \beta^2 + 1}{2|\alpha|} \quad (5) \]

holds.

Čermák J, Tomášek P: On delay-dependent stability conditions for a three-term linear difference equation.

Numerical scheme

We consider

\[ y'(t) = ay(t) + by(t - \tau), \quad t > 0, \]

and

\[ Y(n+2) - \frac{1 + ah}{1 - ah} Y(n) - \frac{2bh}{1 - ah} Y(n-k+1) = 0, \quad n = 0, 1, \ldots \quad (NS) \]

- equidistant mesh: \( t_n = nh, \ n = 0, 1 \ldots \)
- stepsize \( h = \tau/k, \) where \( k \geq 2, \ k \in \mathbb{Z} \)
- \( ah \neq 1. \)
- \( Y(n) \approx y(t_n), \ n = 0, 1, 2, \ldots \)

Remark

Such efficient choice of stepsize makes the discretization formulae free of extra interpolation terms, which can arise from an appropriate approximation of the delayed term.
\[ Y(n + 2) - \frac{1 + ah}{1 - ah} Y(n) - \frac{2bh}{1 - ah} Y(n - k + 1) = 0, \quad n = 0, 1, \ldots \tag{NS} \]

**Theorem**

Let \( k \geq 2 \) be even. Then (NS) is asymptotically stable if and only if one of the following conditions holds

\[
|bh| \leq 1, \quad |b| + a < 0, \quad 2 < 2b^2 h^2 < 1 - ah, \quad \tau < \tau_1^*(h). \tag{6} \tag{7}
\]

where

\[
\tau_1^*(h) = h + 2h \frac{a + b^2 h}{(1 + ah)|b|} \frac{1 + a^2 h^2 - 2b^2 h^2}{a^2 h^2 - 1}. 
\]
Asymptotic stability region in the case of $k$ even
\begin{equation}
Y(n + 2) - \frac{1 + ah}{1 - ah} Y(n) - \frac{2bh}{1 - ah} Y(n - k + 1) = 0, \quad n = 0, 1, \ldots \tag{NS}
\end{equation}

**Theorem**

Let \( k \geq 3 \) be odd and \( m = (k - 1)/2 \). Then (NS) is asymptotically stable if and only if one of the following conditions holds

\[
\begin{align*}
a &\leq b < -a, & |bh| &< 1, \quad (8) \\
|b| + a &< 0, \quad (-1)^m bh = 1, \quad (9) \\
b + |a| &< 0, \quad bh > -1, \quad \tau < \tau_2^*(h), \quad (10) \\
(-1)^m b + a &< 0, \quad (-1)^m bh > 1, \quad \tau < \tau_2^*(h), \quad (11) \\
(-1)^m b + a &> 0, \quad (-1)^{m+1} bh > 1, \quad \tau < \tau_2^*(h). \quad (12)
\end{align*}
\]

where

\[
\tau_2^*(h) = h + 2h \arccos \frac{a + b^2h}{|(1 + ah)b|} / \arccos \frac{1 + a^2h^2 - 2b^2h^2}{|a^2h^2 - 1|}.
\]
Asymptotic stability region in the case of $k$ odd and $m = \frac{k-1}{2}$ even

Figure: $k$ odd and $m = \frac{k-1}{2}$ even
Asymptotic stability region in the case of $k$ odd and $m = \frac{k-1}{2}$ odd

**Figure:** $k$ odd and $m = \frac{k-1}{2}$ odd
We consider the initial value problem for (DDE) with $\tau = 1$

\[ y'(t) = ay(t) + by(t - 1), \quad t > 0, \]
\[ y(t) = 1 \quad \text{for} \quad t \in [-1, 0] \]

and we decide to use formula (NS) to obtain numerical solution.
Example

We consider the initial value problem for (DDE) with $\tau = 1$

$$y'(t) = 30y(t) + \frac{-51}{10} y(t - 1), \quad t \geq 0,$$  \hspace{1cm} (15)

$$y(t) = 1 \quad \text{for} \quad t \in [-1, 0].$$ \hspace{1cm} (16)

**Figure:** AS DDE, numerical solutions for $k \in \{3, 4, 5, 6, 7\}$
Example

We consider the initial value problem for (DDE) with $\tau = 1$

\[
y'(t) = 30y(t) + \frac{-51}{10}y(t-1), \quad t \geq 0, \tag{17}
\]

\[
y(t) = 1 \quad \text{for} \quad t \in [-1, 0]. \tag{18}
\]

**Figure:** $(a, b) = (30, -5.1), \ k = 5$
Example

We consider the initial value problem for (DDE) with $\tau = 1$

$$y'(t) = -y(t) - \frac{3}{2}y(t - 1), \quad t \geq 0,$$

$$y(t) = 1 \quad \text{for} \quad t \in [-1, 0].$$

Figure: $(a, b) = (-1, -1.5)$, $k = 50$; $k = 51$
Figure: \((a, b) = (-1, -1.5), k = 52; \quad k = 53\)
Figure: \((a, b) = (-\frac{1}{\pi}, -\frac{1}{\pi}, 5), k = 50; k = 51, m = 25; k = 52; k = 53, m = 26\)
What happens if $h \to 0^+$?

In the case of $k$ even the asymptotic stability region of (NS) becomes $|b| + a < 0$. With the exception of the boundary, this region corresponds to (1).
What happens if $h \to 0^+$?

In the case of $k$ odd it may be shown (by the L’Hospital rule) that the asymptotic stability conditions turn to

$$a \leq b < -a,$$

$$\left|a\right| + b < 0, \quad \tau < \frac{\arccos(-a/b)}{(b^2-a^2)^{1/2}}$$

as $h \to 0^+$. These are equivalent to the conditions defining the asymptotic stability region of (DDE).
Summary

- Formulation of necessary and sufficient conditions for numerical scheme applied to (DDE)
- Numerical examples - explanation for "unexpected" computation results.
The results are based on ...


- MATLAB and MAPLE computations