Integrability of natural Hamiltonian systems in 2D curved spaces

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Introduction

- Let $H : M \to \mathbb{R}$ be a smooth scalar called Hamiltonian, and

\[
\begin{align*}
\frac{d}{dt} q &= \frac{\partial H}{\partial p}, \\
\frac{d}{dt} p &= -\frac{\partial H}{\partial q},
\end{align*}
\]  

(1)

the associated equations of motion.

- Introducing $x = (q, p)^T$, we can rewrite (1) as

\[
\frac{d}{dt} x = \nu_H(x), \quad \nu_H(x) = \mathbb{I}_n \nabla x H, \quad \mathbb{I}_n = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}.
\]

(2)

- **Question:** How to find all solutions?

\[
x(t) = \varphi(t, x_0), \quad x(0) = x_0.
\]
First integrals help

\[ \frac{d}{dt} x = \nu_H(x), \quad \nu_H(x) = I_{2n} \nabla x H, \quad (3) \]

**Definition**

A non-constant function \( F(x) : M \to \mathbb{R} \) is called a first integral of (3) if \( F(x(t)) = \text{const} \) for all solutions \( x(t) \).

\[ \frac{d}{dt} F(x) = \left( \frac{\partial F}{\partial x} \right)^T \nu_H(x) = \{F, H\}(x) = 0. \]

**Theorem (Liouville)**

*If the Hamiltonian system with \( n - d.o.f. \) has \( n \) functionally independent first integrals which commute, i.e., \( \{F_i, F_j\} = 0 \), for every \( i, j = 1, \ldots, n \), then the equations of motion are integrable by quadrature.*

**Question:** How to hunt for first integrals?
A particular solution and variational equations help!

Let $H : \mathbb{C}^{2n} \to \mathbb{C}$ be a holomorphic Hamiltonian, and

$$\frac{d}{dt} = \nu_H(x), \quad \nu_H(x) = i_{2n} \nabla_x H, \quad x \in \mathbb{C}^{2n}, \quad t \in \mathbb{C}, \quad (4)$$

the associated Hamilton equations.

Let $t \to \phi(t) \in \mathbb{C}^{2n}$ be a non-equilibrium solution of (4).

The maximal analytic continuation of $\phi(t)$ defines a Riemann surface $\Gamma$ with $t$ as a local coordinate.

$$\Gamma := \{ x \in \mathbb{C}^{2n} | x = \phi(t), \ t \in U \in \mathbb{C} \}.$$

Variational equations along $\phi(t)$ have the form

$$\frac{d}{dt} \xi = A(t) \cdot \xi, \quad A(t) = \frac{\partial \nu_H}{\partial x}(\phi(t)). \quad (5)$$

We can attach to the equation (5) the differential Galois group $G$. 
Morales-Ramis theorem

**Theorem**

Assume that a Hamiltonian system is meromorphically integrable in the Liouville sense in a neighbourhood of the analytic phase curve $\Gamma$. Then the identity component of the differential Galois group of the variational equations along $\Gamma$ is Abelian.


Applications of Morales–Ramis theory

- to prove non-integrability of Hamiltonian systems,


- to detection possible integrable cases for Hamiltonian systems depending on parameters.


Main steps during applications

- Find a particular solution different from equilibrium points,
- calculate VE and NVE,
- check if $G^0$ is Abelian (most difficult step): we try to transform NVE into the equation with known differential Galois group:
  - Riemann $P$ equation,
  - Lamé equation,
  - an equation of the second order with rational coefficients.

Integrability of homogeneous Hamiltonian equations

Integrability of Hamiltonian systems given by

\[ H = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + V(q), \quad (q, p) \in \mathbb{C}^{2n}, \]

\( V \) — homogeneous of degree \( k \in \mathbb{Z} \)

\[ V(\lambda q_1, \ldots, \lambda q_n) = \lambda^k V(q_1, \ldots, q_n) \]

**Definition (standard)**

Darboux point \( d \in \mathbb{C}^n \) is a non-zero solution of

\[ V'(d) = d \]

Particular solution

\[ q(t) = \varphi(t)d, \quad p(t) = \dot{\varphi}(t)d \quad \text{provided} \quad \ddot{\varphi} = -\varphi^{k-1}. \]
Integrability of homogeneous Hamiltonian equations

On the energy level:

$$H(\varphi(t)\mathbf{d}, \dot{\varphi}(t)\mathbf{d}) = e \in \mathbb{C}^*,$$

hyperelliptic curve

$$\dot{\varphi}^2 = \frac{2}{k} \left( \varepsilon - \varphi^k \right), \quad \varepsilon = ke \in \mathbb{C}^*.$$

The variational equations

$$\ddot{x} = -\lambda \varphi(t)^{k-2} x,$$

where $\lambda$ is an eigenvalue of $V''(\mathbf{d})$.

What is analog of homogeneous systems in curved spaces?

No obvious answer

\[ H = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + V(q), \quad (q, p) \in \mathbb{C}^{2n}, \]

Our first proposition

\[ H = \frac{1}{2} r^{m-k} \left( p_r^2 + \frac{p_\varphi^2}{r^2} \right) + r^m U(\varphi), \]

where \( m \) and \( k \) are integers, and \( k \neq 0 \).

We obtain obstructions on values of the quantities\(^1\)

\[ \lambda = 1 + \frac{U''(\varphi_0)}{kU(\varphi_0)}, \quad \text{where} \quad U'(\varphi_0) = 0. \quad (7) \]

\[ U(\varphi) = -\cos \varphi. \] Superintegrable cases

- **Case 1:** \( m = 1, \ k = -5. \)

\[
H = \frac{1}{2} r^6 \left( p_r^2 + \frac{p_{\varphi}^2}{r^2} \right) - r \cos \varphi,
\]

\[
F_1 := r^2 p_{\varphi}^2 \cos(2\varphi) - r^{-1} p_r p_{\varphi} \sin(2\varphi) + r^{-1} \sin \varphi \sin(2\varphi),
\]

\[
F_2 := r^2 p_{\varphi}^2 \sin(2\varphi) + r^{-1} p_r p_{\varphi} \cos(2\varphi) - r^{-1} \sin \varphi \cos(2\varphi).
\]

- **Case 2:** \( m = -1, \ k = 1. \)

\[
H = \frac{1}{2} r^{-2} \left( p_r^2 + \frac{p_{\varphi}^2}{r^2} \right) - r^{-1} \cos \varphi,
\]

\[
F_1 := r^{-2} p_{\varphi}^2 \cos(2\varphi) + r^{-1} p_r p_{\varphi} \sin(2\varphi) + r \sin \varphi \sin(2\varphi),
\]

\[
F_2 := -r^{-2} p_{\varphi}^2 \sin(2\varphi) + r^{-1} p_r p_{\varphi} \cos(2\varphi) + r \sin \varphi \cos(2\varphi).
\]
Homogeneous potentials in curved spaces

$U(\varphi) = -\cos \varphi$. Super-integrable cases

- **Case 3:** $m = 1$, $k = 1$.

  \[ H = \frac{1}{2} \left( p_r^2 + \frac{p_\varphi^2}{r^2} \right) - r \cos \varphi, \]

  \[ F_1 := r^{-1} p_\varphi^2 \cos \varphi + p_r p_\varphi \sin \varphi + \frac{1}{2} r^2 \sin^2 \varphi, \]

  \[ F_2 := \left( p_r^2 - r^{-2} p_\varphi^2 \right) \cos \varphi \sin \varphi + r^{-1} p_r p_\varphi \cos(2\varphi) - r \sin \varphi. \]

- **Case 4:** $m = -1$, $k = -5$.

  \[ H = \frac{1}{2} r^4 \left( p_r^2 + \frac{p_\varphi^2}{r^2} \right) - r^{-1} \cos \varphi, \]

  \[ F_1 := r p_\varphi^2 \cos \varphi - r^2 p_r p_\varphi \sin \varphi + \frac{1}{2} r^{-2} \sin^2 \varphi, \]

  \[ F_2 := r^4 \left( p_r^2 - r^{-2} p_\varphi^2 \right) \cos \varphi \sin \varphi - r^3 p_r p_\varphi \cos(2\varphi) - r^{-1} \sin \varphi. \]
Higher order first integrals

- $m = 3(k + 2), \quad U(\varphi) = \cosh(\sqrt{3}(k + 2)\varphi)$

\[ H = \frac{1}{2} \left( p_r^2 + \frac{p_\varphi^2}{r^2} \right) r^{2(k+3)} + r^{3(k+2)} \cosh(\sqrt{3}(k + 2)\varphi). \quad (8) \]

Cubic first integral

\[ F = (k + 2) p_\varphi^3 - \frac{3}{4} r^{3+k} U'(\varphi) p_r + \frac{9}{4} (k + 2) r^{k-2} U(\varphi) p_\varphi. \quad (9) \]

- $m = 2k, \quad U(\varphi) = \cosh((k + 2)\varphi)$

\[ H = \frac{1}{2} \left( p_r^2 + \frac{p_\varphi^2}{r^2} \right) r^2 + r^{(k+2)} \cosh((k + 2)\varphi). \quad (10) \]

Quartic first integral

\[ F = r^2 (k + 2)^2 p_r^2 p_\varphi^2 + 2(k + 2) r^{k+2} U'(\varphi) p_r p_\varphi + r^{2(k+2)} U'(\varphi)^2. \quad (11) \]
Another analogue in curved spaces

Our second proposition

\[ H = \frac{1}{2} \left( p_r^2 + \frac{p_{\varphi}^2}{S_\kappa(r)^2} \right) + S_\kappa(r)^m U(\varphi), \]  

(12)

where \( m \in \mathbb{Z} \) and \( U(\varphi) \) is a meromorphic function and \( S_\kappa(r) \) is defined by

\[ S_\kappa(r) := \begin{cases} 
\frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa}r) & \text{for } \kappa > 0, \\
r & \text{for } \kappa = 0, \\
\frac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa}r) & \text{for } \kappa < 0. 
\end{cases} \]  

(13)

We obtain obstructions on values of the quantities\(^2\)

\[ \lambda = 1 + \frac{U''(\varphi_0)}{kU(\varphi_0)}, \quad \text{where} \quad U'(\varphi_0) = 0. \]  

(14)

Integrable cases and super-integrable cases

- \( U(\varphi) = \cos^k \varphi \) and \( k \)-arbitrary

\[
H = \frac{1}{2} \left( p_r^2 + \frac{p_{\varphi}^2}{S_\kappa(r)^2} \right) + S_\kappa^m(r) \cos^m \varphi,
\]

(15)

Linear first integral

\[
l_\kappa = p_r \sin \varphi + p_{\varphi} \cos \varphi \sqrt{\kappa} \cot \sqrt{\kappa} r, \quad \kappa \neq 0.
\]

(16)

Limit

\[
l_0 = \lim_{\kappa \to 0} l_\kappa = p_r \sin \varphi + r^{-1} p_{\varphi} \cos \varphi,
\]

(17)

which gives the first integral for the case \( \kappa = 0 \).

- \( U(\varphi) = \cos \varphi, \kappa = 0 \) and \( k = 1 \), then there exists additional independent first integral quadratic in momenta

\[
l_2 = \left( p_r^2 - \frac{p_{\varphi}^2}{r^2} \right) \cos \varphi \sin \varphi + r^{-1} p_r p_{\varphi} \cos(2\varphi) - r \sin \varphi.
\]

(18)

Thus, in this case the system is maximally super-integrable.
Integrable cases and super-integrable cases

- \( U(\varphi) = \cos(2\varphi) \) and \( k = 2 \).

\[
H = \frac{1}{2} \left( p_r^2 + \frac{p_{\varphi}^2}{S_\kappa(r)^2} \right) + S_\kappa(r)^2 \cos(2\varphi) \tag{19}
\]

Quadratic first integral

\[
I = \left[ p_r^2 - \left( p_{\varphi} \frac{C_\kappa(r)}{S_\kappa(r)} \right)^2 \right] U(\varphi) + p_r p_{\varphi} \frac{C_\kappa(r)}{S_\kappa(r)} U'(\varphi) + 2(c_1^2 + c_2^2) S_\kappa(r)^2 \tag{20}
\]
What about an arbitrary form of the metric?

- First system

\[ H = \frac{1}{2} r^{m-k} \left( p_r^2 + \frac{p_\phi^2}{r^2} \right) + r^m U(\varphi), \]

- Second system

\[ H = \frac{1}{2} \left( p_r^2 + \frac{p_\phi^2}{S_\kappa(r)^2} \right) + S_\kappa(r)^m U(\varphi), \]

- When \( \kappa = 0 \), then \( M^2 = \mathbb{R}^2 \) is a Cartesian plane
- When \( \kappa > 0 \), then \( M^2 = S^2 \) is a sphere
- When \( \kappa < 0 \), then \( M^2 = \mathbb{H}^2 \) is a hyperbolic plane.
What about an arbitrary form of the metric?

■ First system

\[ H = \frac{1}{2} r^{m-k} \left( p_r^2 + \frac{p_\varphi^2}{r^2} \right) + r^m U(\varphi), \]

■ Second system

\[ H = \frac{1}{2} \left( p_r^2 + \frac{p_\varphi^2}{S_\kappa(r)^2} \right) + S_\kappa(r)^m U(\varphi), \]

■ When \( \kappa = 0 \), then \( M^2 = \mathbb{R}^2 \) is a Cartesian plane
■ When \( \kappa > 0 \), then \( M^2 = S^2 \) is a sphere
■ When \( \kappa < 0 \), then \( M^2 = \mathbb{H}^2 \) is a hyperbolic plane.

Our third proposition

\[ H = \frac{1}{2} \left( a(r)p_r^2 + b(r)p_\varphi^2 \right) + c(r)\cos \varphi + d(r)\sin \varphi, \quad (21) \]

where \( a(r), b(r), c(r) \) and \( d(r) \) are meromorphic functions of variable \( r \).
Main integrability theorem. Auxiliary sets

\[ M_1(\mu) := \left\{ \frac{1}{4} \left( 1 + 4p \right) \left( 1 + 4p \pm \sqrt{1 + 8\mu} \right) \mid p \in \mathbb{Z} \right\}, \quad (22) \]

\[ M_2(\mu) := \left\{ \left( p + \frac{1}{2} \right)^2 - \left( \mu + \frac{1}{4} \right)^2 + \mu^2 \mid p \in \mathbb{Z} \right\}, \quad (23) \]

\[ M_3(\mu) := \left\{ \left( p + \frac{1}{3} \right)^2 - \left( \mu + \frac{1}{4} \right)^2 + \mu^2 \mid p \in \mathbb{Z} \right\}, \quad (24) \]

\[ M_4(\mu) := \left\{ \left( p + \frac{1}{4} \right)^2 - \left( \mu + \frac{1}{4} \right)^2 + \mu^2 \mid p \in \mathbb{Z} \right\}, \quad (25) \]

\[ M_5(\mu) := \left\{ \left( p + \frac{1}{5} \right)^2 - \left( \mu + \frac{1}{4} \right)^2 + \mu^2 \mid p \in \mathbb{Z} \right\}, \quad (26) \]

\[ M_6(\mu) := \left\{ \left( p + \frac{2}{5} \right)^2 - \left( \mu + \frac{1}{4} \right)^2 + \mu^2 \mid p \in \mathbb{Z} \right\}. \quad (27) \]
Theorem (Main Theorem)

Assume that \( a(r) \), \( b(r) \), \( c(r) \) and \( d(r) \) are meromorphic functions and there exists a point \( r_0 \in \mathbb{Z} \) such that

\[
\begin{align*}
  b'(r_0) &= c'(r_0) = d'(r_0) = 0, \quad b(r_0) \neq 0, \quad \text{and} \quad c(r_0) \neq -id(r_0). \quad (28)
\end{align*}
\]

If the Hamiltonian system defined by the Hamiltonian

\[
H = \frac{1}{2} \left( a(r)p_r^2 + b(r)p_\varphi^2 \right) + c(r)\cos \varphi + d(r)\sin \varphi, \quad (29)
\]

is integrable in the Liouville sense, then the numbers

\[
\begin{align*}
  \mu &:= \frac{a(r_0)((c(r_0) + id(r_0))b''(r_0) - b(r_0)(c''(r_0) + id''(r_0)))}{b(r_0)^2(c(r_0) + id(r_0))}, \\
  \lambda &:= i \frac{a(r_0)(c(r_0)d''(r_0) - d(r_0)c''(r_0))}{b(r_0)(c(r_0)^2 + d(r_0)^2)}, \quad (30)
\end{align*}
\]

belong to the following table.
# Integrability Table

<table>
<thead>
<tr>
<th>No.</th>
<th>$\mu$</th>
<th>$\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\mathbb{C}$</td>
<td>$\mathcal{M}_1(\mu) \cup \mathcal{M}_2(\mu)$</td>
</tr>
<tr>
<td>2</td>
<td>$2 \left( q + \frac{1}{2} \right)^2 - \frac{1}{8}$</td>
<td>$\mathbb{C}$</td>
</tr>
<tr>
<td>3</td>
<td>$2q^2 + q$</td>
<td>$\mathcal{M}_3(\mu)$</td>
</tr>
<tr>
<td>4</td>
<td>$2 \left( q + \frac{1}{3} \right)^2 - \frac{1}{8}$</td>
<td>$\bigcup_{i=3}^{6} \mathcal{M}_i(\mu)$</td>
</tr>
<tr>
<td>5</td>
<td>$2 \left( q + \frac{1}{5} \right)^2 - \frac{1}{8}$</td>
<td>$\mathcal{M}_3(\mu) \cup \mathcal{M}_6(\mu)$</td>
</tr>
<tr>
<td>6</td>
<td>$2 \left( q + \frac{2}{5} \right)^2 - \frac{1}{8}$</td>
<td>$\mathcal{M}_3(\mu) \cup \mathcal{M}_5(\mu)$</td>
</tr>
</tbody>
</table>

**Table:** Integrability table. Here $q \in \mathbb{Z}$ and the sets $\mathcal{M}_i(\mu)$ are defined in (42)–(47).
Outline of the proof. Vector field

The system

\[
\begin{align*}
\dot{r} &= \frac{\partial H}{\partial p_r} = a(r)p_r, \\
\dot{p}_r &= -\frac{\partial H}{\partial r} = -\frac{1}{2} \left( a'(r)p_r^2 + b'(r)p_\varphi^2 \right) - c'(r)\cos \varphi - d'(r)\sin \varphi, \\
\dot{\varphi} &= \frac{\partial H}{\partial p_\varphi} = b(r)p_\varphi, \\
\dot{p}_\varphi &= -\frac{\partial H}{\partial \varphi} = c(r)\sin \varphi - d(r)\cos \varphi.
\end{align*}
\]

(31)

If \( b'(r_0) = c'(r_0) = d'(r_0) = 0 \), for a certain \( r_0 \in \mathbb{C} \), then the system (31) possesses the invariant manifold

\[
\mathcal{N} = \left\{ (r, p_r, \varphi, p_\varphi) \in \mathbb{C}^4 \mid r = r_0, \ p_r = 0 \right\},
\]

(32)

and its restriction to \( \mathcal{N} \) is given by

\[
\begin{align*}
\dot{r} &= \dot{p}_r = 0, \\
\dot{\varphi} &= b(r_0)p_\varphi, \\
\dot{p}_\varphi &= c(r_0)\sin \varphi - d(r_0)\cos \varphi.
\end{align*}
\]

(33)

\[
\dot{\varphi}^2 = 2b(r_0) \left\{ E - c(r_0)\cos \varphi - d(r_0)\sin \varphi \right\}.
\]

(34)
Outline of the proof. Variational equations

- Particular solution
  
  \[ \varphi(t) = (0, 0, \varphi(t)p_\varphi(t)) \].

- The first order variational equations along \( \varphi(t) \):

  \[
  \frac{d}{dt} X = A(t)X, \quad A(t) = \frac{\partial v_H(x)}{\partial x}(\varphi(t)),
  \tag{35}
  \]

  where the matrix \( A(t) \) has the form

  \[
  A(t) = \begin{bmatrix}
  0 & \cdots & a(r_0) & 0 & 0 \\
  \Xi - E & \cdots & b''(r_0) & -c''(r_0)\cos \varphi - d''(r_0)\sin \varphi & 0 & 0 & 0 \\
  0 & \cdots & 0 & 0 & b(r_0) \\
  0 & \cdots & 0 & 0 & \Xi \\
  0 & \cdots & 0 & 0 & 0 \\
  \end{bmatrix}
  \]

  \[ \Xi := c(r_0)\cos \varphi + d(r_0)\sin \varphi. \]

  \( X = [R, P_R, \Phi, P_\Phi]^T \) denotes the variations of \( x = [r, p_r, \varphi, p_\varphi]^T \).
Homogeneous potentials in curved spaces

Outline of the proof. Rationalization

► The normal part

\[
\begin{pmatrix}
\dot{R} \\
\dot{P}_R
\end{pmatrix} =
\begin{pmatrix}
0 & a(r_0) \\
(\Xi - E) \frac{b''(r_0)}{b(r_0)} - c''(r_0)\cos \varphi - d'''(r_0)\sin \varphi & 0
\end{pmatrix}
\begin{pmatrix}
R \\
P_R
\end{pmatrix}
\] (36)

can be rewritten as a one second-order differential equation

\[
\ddot{R} = a(r_0) \left( (c(r_0)\cos \varphi + d(r_0) - E\sin \varphi) \frac{b''(r_0)}{b(r_0)} - c''(r_0)\cos \varphi - d'''(r_0)\sin \varphi \right) R.
\]

► Change of independent variable

\[
t \longrightarrow z := e^{2i\varphi(t)} \left( 1 - \frac{2c(r_0)}{c(r_0) + id(r_0)} \right)
\] (37)

on the level \( E = 0 \), transforms NVE into

\[
\frac{d^2R}{dz^2} + \left( \frac{3}{4z} + \frac{1}{2(z - 1)} \right) \frac{dR}{dz} - \left( \frac{\mu}{8z^2} + \frac{\lambda}{4z(z - 1)} \right) R = 0,
\] (38)
Outline of the proof. Rationalization

\[ \frac{d^2 R}{dz^2} + \left( \frac{3}{4z} + \frac{1}{2(z-1)} \right) \frac{dR}{dz} - \left( \frac{\mu}{8z^2} + \frac{\lambda}{4z(z-1)} \right) R = 0, \quad (39) \]

where

\[ \mu := \frac{a(r_0)((c(r_0) + id(r_0))b''(r_0) - b(r_0)(c''(r_0) + id''(r_0)))}{b(r_0)^2(c(r_0) + id(r_0))}, \]

\[ \lambda := i \frac{a(r_0)(c(r_0)d''(r_0) - d(r_0)c''(r_0))}{b(r_0)(c(r_0)^2 + d(r_0)^2)}, \]
Outline of the proof. Rationalization

$$\frac{d^2 R}{dz^2} + \left( \frac{3}{4z} + \frac{1}{2(z-1)} \right) \frac{dR}{dz} - \left( \frac{\mu}{8z^2} + \frac{\lambda}{4z(z-1)} \right) R = 0, \quad (39)$$

where

$$\mu : = \frac{a(r_0)((c(r_0) + id(r_0))b''(r_0) - b(r_0)(c''(r_0) + id''(r_0))}{b(r_0)^2(c(r_0) + id(r_0))},$$

$$\lambda : = i \frac{a(r_0)(c(r_0)d''(r_0) - d(r_0)c''(r_0))}{b(r_0)(c(r_0)^2 + d(r_0)^2)},$$

► Form of the Riemman $P$ equation

$$R'' + \left( \frac{1 - \alpha - \alpha'}{z} + \frac{1 - \gamma - \gamma'}{z - 1} \right) R' + \left( \frac{\alpha \alpha'}{z^2} + \frac{\gamma \gamma'}{(z - 1)^2} + \frac{\beta \beta' - \alpha \alpha' - \gamma \gamma'}{z(z - 1)} \right) R = 0,$$

The differences of exponents at singularities $z = 0, z = 1$ and $z = \infty$

$$\rho = \alpha - \alpha' = \frac{\sqrt{\Delta^2 - 16\lambda}}{4}, \quad \sigma = \gamma - \gamma' = \frac{1}{2}, \quad \tau = \beta - \beta' = \frac{\Delta}{4}, \quad (40)$$

where

$$\Delta = \sqrt{1 + 16\lambda + 8\mu}.$$
Solvability of Riemann P equation. Kimura theorem

Theorem (Kimura)

The identity component of the differential Galois group of the Riemann P equation is solvable iff

A. at least one of the four numbers $\rho + \sigma + \tau$, $-\rho + \sigma + \tau$, $\rho - \sigma + \tau$, $\rho + \sigma - \tau$ is an odd integer, or

B. the numbers $\rho$ or $-\rho$ and $\sigma$ or $-\sigma$ and $\tau$ or $-\tau$ belong (in an arbitrary order) to some of appropriate fifteen families forming the so-called Schwarz’s table fifteen families.
where \( l, s, q \in \mathbb{Z} \).
Kimura theorem: Condition A

The case A of the Kimura Theorem is satisfied if and only if one of the numbers

\[
\rho + \sigma + \tau = \frac{1}{4} \left( 2 + \Delta + \sqrt{\Delta^2 - 16\lambda} \right),
\]

\[
-\rho + \sigma + \tau = \frac{1}{4} \left( 2 + \Delta - \sqrt{\Delta^2 - 16\lambda} \right),
\]

\[
\rho - \sigma + \tau = \frac{1}{4} \left( -2 + \Delta + \sqrt{\Delta^2 - 16\lambda} \right),
\]

\[
\rho + \sigma - \tau = \frac{1}{4} \left( 2 - \Delta + \sqrt{\Delta^2 - 16\lambda} \right)
\]
is an odd integer. It is easy to check that if one the above numbers is an odd integer, then \( \lambda \in M_1(\mu) \), where

\[
M_1(\mu) = \left\{ \frac{1}{4} \left( 1 + 4p \right) \left( 1 + 4p \pm \sqrt{\Delta^2 - 16\lambda} \right) \mid p \in \mathbb{Z} \right\}
\]
Kimura Theorem: Condition B

In this case the quantities $\rho$ or $-\rho$, $\sigma$ or $-\sigma$ and $\tau$ or $-\tau$ must belong to Schwarz's table. As $\sigma = \frac{1}{2}$ only items 1, 2, 4, 6, 9, or 14 are allowed.

Case 1.

- $\pm \rho = 1/2 + q$, for a certain $q \in \mathbb{Z}$, then $\mu = 2 \left( q + \frac{1}{2} \right)^2 - \frac{1}{8}$. In this case $\tau$ is arbitrary, and thus $\lambda$ is arbitrary.
- $\pm \tau = 1/2 + p$, for certain $p \in \mathbb{Z}$, then $\lambda \in \mathcal{M}_2(\mu)$. In this case $\rho$ is arbitrary, so $\mu$ is arbitrary.

Case 2. In this case $\pm \rho = 1/3 + q$ and $\pm \tau = 1/3 + p$, for certain $q, p \in \mathbb{Z}$. These conditions imply that $\lambda \in \mathcal{M}_3(\mu)$, and

$$\mu = 2 \left( q + \frac{1}{3} \right)^2 - \frac{1}{8}. \quad (41)$$

Case 4.

- $\pm \rho = 1/3 + q$, and $\pm \tau = 1/4 + p$, for certain $q, p \in \mathbb{Z}$, then $\lambda \in \mathcal{M}_4(\mu)$ and $\mu$ is given by (41).
- $\pm \rho = 1/4 + q$, and $\pm \tau = 1/3 + p$, for certain $q, p \in \mathbb{Z}$, then $\lambda \in \mathcal{M}_3(\mu)$ and $\mu = 2q^2 + q$. 

[8x266]Homogeneous potentials in curved spaces

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28 / 43
Integrability Table

<table>
<thead>
<tr>
<th>No.</th>
<th>( \mu )</th>
<th>( \lambda )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \mathbb{C} )</td>
<td>( \mathcal{M}_1(\mu) \cup \mathcal{M}_2(\mu) )</td>
</tr>
<tr>
<td>2</td>
<td>( 2 \left( q + \frac{1}{2} \right)^2 - \frac{1}{8} ) ( \mathbb{C} )</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>( 2q^2 + q )</td>
<td>( \mathcal{M}_3(\mu) )</td>
</tr>
<tr>
<td>4</td>
<td>( 2 \left( q + \frac{1}{3} \right)^2 - \frac{1}{8} ) ( \bigcup_{i=3}^{6} \mathcal{M}_i(\mu) )</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>( 2 \left( q + \frac{1}{5} \right)^2 - \frac{1}{8} ) ( \mathcal{M}_3(\mu) \cup \mathcal{M}_6(\mu) )</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>( 2 \left( q + \frac{2}{5} \right)^2 - \frac{1}{8} ) ( \mathcal{M}_3(\mu) \cup \mathcal{M}_5(\mu) )</td>
<td></td>
</tr>
</tbody>
</table>

Table: Integrability table. Here \( q \in \mathbb{Z} \) and the sets \( \mathcal{M}_i(\mu) \) are defined in (42)–(47).
Main integrability theorem. Auxiliary sets

\[ M_1(\mu) := \left\{ \frac{1}{4} (1 + 4p) \left( 1 + 4p \pm \sqrt{1 + 8\mu} \right) \mid p \in \mathbb{Z} \right\}, \quad (42) \]

\[ M_2(\mu) := \left\{ \left( p + \frac{1}{2} \right)^2 - \left( \mu + \frac{1}{4} \right)^2 + \mu^2 \mid p \in \mathbb{Z} \right\}, \quad (43) \]

\[ M_3(\mu) := \left\{ \left( p + \frac{1}{3} \right)^2 - \left( \mu + \frac{1}{4} \right)^2 + \mu^2 \mid p \in \mathbb{Z} \right\}, \quad (44) \]

\[ M_4(\mu) := \left\{ \left( p + \frac{1}{4} \right)^2 - \left( \mu + \frac{1}{4} \right)^2 + \mu^2 \mid p \in \mathbb{Z} \right\}, \quad (45) \]

\[ M_5(\mu) := \left\{ \left( p + \frac{1}{5} \right)^2 - \left( \mu + \frac{1}{4} \right)^2 + \mu^2 \mid p \in \mathbb{Z} \right\}, \quad (46) \]

\[ M_6(\mu) := \left\{ \left( p + \frac{2}{5} \right)^2 - \left( \mu + \frac{1}{4} \right)^2 + \mu^2 \mid p \in \mathbb{Z} \right\}. \quad (47) \]
Theorem (Main Theorem)

Assume that $a(r)$, $b(r)$, $c(r)$ and $d(r)$ are meromorphic functions and there exists a point $r_0 \in \mathbb{Z}$ such that

$$b'(r_0) = c'(r_0) = d'(r_0) = 0, \quad b(r_0) \neq 0, \quad \text{and} \quad c(r_0) \neq -id(r_0). \quad (48)$$

If the Hamiltonian system defined by the Hamiltonian

$$H = \frac{1}{2} \left( a(r)p_r^2 + b(r)p_{\varphi}^2 \right) + c(r)\cos \varphi + d(r)\sin \varphi,$$

is integrable in the Liouville sense, then the numbers

$$\mu := \frac{a(r_0)((c(r_0) + id(r_0))b''(r_0) - b(r_0)(c''(r_0) + id''(r_0)))}{b(r_0)^2(c(r_0) + id(r_0))},$$

$$\lambda := i \frac{a(r_0)(c(r_0)d''(r_0) - d(r_0)c''(r_0))}{b(r_0)(c(r_0)^2 + d(r_0)^2)},$$

belong to the Table 2.
Application of the Theorem 2. First example

Let us consider the following Hamiltonian function

\[ H = \frac{1}{2} \left\{ p_r^2 + \left( n + \sin^{-2} r \right) p_{\phi}^2 \right\} + \sin r \cos \varphi, \]  

(51)

with \( n \in \mathbb{Z} \).

The functions \( a, b, c, d \) are

\[ a(r) = 1, \quad b(r) = n + \sin^{-2} r, \quad c(r) = \sin r, \quad d(r) = 0. \]  

(52)

We take a point \( r_0 = \pi/2 \), at which the condition (48) is fulfilled.

The values of \( \mu \) and \( \lambda \) at \( r_0 \) are given by

\[ \mu = \frac{3 + n}{(1 + n)^2}, \quad \lambda = 0. \]  

(53)

Possibly integrable cases \( n \in \{0, 1, 3, -3, -2, 11\} \).
First example. Integrable cases

1. For \( n = 0 \), the system (51) possesses linear first integral

\[
F = p_r \sin \varphi + p_\varphi \cot r \cos \varphi. \tag{54}
\]

2. For \( n = 1 \), the Hamiltonian (51) coincide with the famous Kovalevskaya case defined on sphere \( S^2 \) that has the quartic first integral

\[
I = p_\varphi^4 \sin^2 r + p_r^2 p_\varphi^2 + 2p_\varphi^2 \sin^{-1} r \cos \varphi + 2p_r p_\varphi \cos r \sin \varphi
+ \frac{1}{4} \left( \cos(2\varphi) + 2 \cos(2r) \sin^2 \varphi \right), \tag{55}
\]

3. For \( n = 3 \), the Hamiltonian (51) corresponds to the Goryachiev–Chaplygin system defined on sphere \( S^2 \) that possesses the following first integrals cubic in momenta

\[
I = p_\varphi^3 \cot^2 r + p_\varphi p_r^2 + p_r \cos r \sin \varphi + p_\varphi \frac{\cos^2 r}{\sin r} \cos \varphi, \tag{56}
\]

...and what about the cases \( n \in \{-3, -2, 11\} \)?
First example. Not integrable cases

Figure: Poincaré section for \( n = -3 \) on the level \( E = 2 \). Cross plane \( r = \pi/2, \ p_r > 0 \)
First example. Not integrable cases

Figure: Poincaré section for $n = -2$ on the level $E = 2$. Cross plane $r = \pi/2$, $p_r > 0$
First example. Not integrable cases

Figure: Poincaré section for $n = 11$ on the level $E = 2$. Cross plane $r = \pi/2$, $p_r > 0$
Application of the Theorem 2. Second example

Let us consider the following Hamiltonian function

\[ H = \frac{1}{2} \left\{ p_r^2 + \left( n^2 + k^2 \sin^{-2} r + \frac{n^2}{4} \tan^2 r \right) p_\varphi^2 \right\} + \sin^k r \cos^n r \cos \varphi, \]  

(57)

where \( k, n \in \mathbb{Z} \).

The functions \( a, b, c, d \) are

\[ a(r) = 1, \quad b(r) = n^2 + k^2 \sin^{-2} r + \frac{n^2}{4} \tan^2 r, \quad c(r) = \sin^k r \cos^n r, \quad d(r) = 0. \]

We take a point \( r_0 = \arccot \left( \sqrt{n/(2k)} \right) \), at which \( b'(r_0) = c'(r_0) = 0 \).

The values of \( \mu \) and \( \lambda \) at \( r_0 \) are given by

\[ \mu = \frac{(2k + n)(k^2 + n(n + 2) + k(n + 4))}{(k^2 + kn + n^2)^2}, \quad \lambda = 0. \]  

(58)

Possibly integrable cases

1. \( n = -4, \) and \( k \in \{4, 8\} \),
2. \( n = -2, \) and \( k \in \{0, -2\} \),
3. \( n = -1, \) and \( k \in \{\pm 1, 2\} \),
4. \( n = 0, \) and \( k \in \{\pm 1, \pm 2, \pm 4, 8, 24\} \),
5. \( n = 1, \) and \( k \in \{0, \pm 1, -2\} \),
6. \( n = 2, \) and \( k \in \{0, \pm 2, \pm 4\} \),
7. \( n = 4, \) and \( k \in \{0, 4\} \).
Second example. Integrable cases

1. For $n = 0, k = -2$ the system (57) is separable with the first integral

$$I = \frac{1}{2} p_\varphi^2 + \cos(2\varphi).$$  

(59)

2. The values $n = 0, k = 1$ correspond to case given in 54.

3. For $n = 0, k = 2$ the system (57) possesses a quadratic first integral

$$I = (p_r^2 - 4p_\varphi^2 \cot^2 r) \cos(\varphi) - 4p_r p_\varphi \cot r \sin(\varphi) - \cos(2r).$$  

(60)

4. For $n = 2, k = 0$ the Hamiltonian (57) corresponds to the integrable Goryachev–Chaplygin system with the first integral given in (56).

5. For $n = -1, k = 1$ the Hamiltonian (57) has a cubic first integral

$$I = \left(4 \sin^{-2} r + \tan^2 r \right) p_\varphi^3 + 4p_r^2 p_\varphi + 8p_r \sqrt{\cos r} \sin \varphi + \frac{2(3 + \cos(2r))}{\sqrt{\cos r} \sin r} p_\varphi \cos \varphi.$$  

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Second example. Not integrable cases

Figure: Poincaré section with $n = 1$, $k = 0$ at the level $E = 3$ on the surface $r = 1$
Second example. Not integrable cases

Figure: Poincaré section with $n = 1, k = -1$ at the level $E = 3$ on the surface $r = 1$
Second example. Not integrable cases

Figure: Poincaré section with $n = 0$, $k = 4$ at the level $E = 3$ on the surface $r = 1$
Summary

Conclusions
- Morales–Ramis theory - the most effective method
- New integrable as well as super-integrable cases were detected

Questions and open problems
- If the necessary integrability conditions are satisfied but it seems that the system is chaotic, how to proof its non-integrability?
- To apply the Main Theorem to the Hamiltonian

\[ H = \frac{1}{2} \left( a(r)p_r^2 + b(r)p_\varphi^2 \right) + c(r)\cos \varphi + d(r)\sin \varphi, \]  

with a more complex form of functions \( a, b, c, d \), and to find new, still unknown integrable cases.
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THANK YOU FOR YOUR ATTENTION!
References


