Systems of ordinary differential equations with nonlocal boundary conditions

Jean Mawhin

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Dear Prof. Ledoux,

Thank you very much for your letter of June 43. I shall welcome with pleasure Mr. Mawhin at our Conference in September. Unfortunately we have no more announcement in English about the Conference available. So I can enclose only an application form.

We could reserve for Mr. Mawhin a room in college in Prague, but we should be scarcely able to guarantee the participation at the sight-seeing tour of Prague on September 7 and at the outing on September 9, if Mr. Mawhin were interested in these events. The closing date for sending the application form was April 30.

Yours sincerely,

S. [Signature]

Dr. Ing. Dmlk, Dr Sc.

Chairman of the Organizing Committee
Systems of ordinary differential equations with nonlocal boundary conditions – p.3/??
Thank you to my Czech colleagues for 50 years of fruitful collaboration and friendship

and thank you to my friends of Brno for 45 years of warm hospitality
A classical existence theorem

\[ \mathbb{R}^n, \langle \cdot | \cdot \rangle, | \cdot |, B_R; \quad f \in C([0, 1] \times \mathbb{R}^n, \mathbb{R}^n) \]

**Thm.** If \( \exists R > 0 : \)

either

\[ \langle u | f(t, u) \rangle \geq 0, \forall (t, u) \in [0, 1] \times \partial B_R, \]

or

\[ \langle u | f(t, u) \rangle \leq 0, \forall (t, u) \in [0, 1] \times \partial B_R, \]

then

\[ x' = f(t, x), \quad x(0) = x(1) \]

has at least one solution taking values in \( \overline{B}_R \)
A classical existence theorem

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\[ x' = f(t, x), \ x(0) = x(1) \]

has at least one solution taking values in \( \overline{B}_R \)

equivalent statements : \( \tau = 1 - t \)

nonlinear version of the linear result :

**Prop.** \( \forall \lambda \in \mathbb{R} \setminus \{0\}, \ \forall e \in C([0, T], \mathbb{R}^n) \)

\[ x' = \lambda x + e(t), \ x(0) = x(1) \] has a solution
\[ \langle u, f(t,u) \rangle \geq 0 \]  \hspace{1cm}  \[ \langle u, f(t,u) \rangle \leq 0 \]
References

special case (not mentioned !) of Theorem 3.2 in M.A. \textsc{Krasnosel’skii}, \textit{The Operator of Translation along the Trajectories of Differential Equations}, Moscow, 1966
Krasnosel’skii’s theorem

Thm. If \( \exists C \), bounded open convex set, \( \Phi_i \in C^1(\mathbb{R}^n, \mathbb{R}) \)
\((i = 1, \ldots, r)\): \( \overline{C} = \{u \in \mathbb{R}^n : \Phi_i(u) \leq 0 \ (i = 1, \ldots, r)\} \),
\( \Phi_i(u) = 0 \) for some \( u \in \partial C \) \( \Rightarrow \nabla \Phi_i(u) \neq 0 \), and
either
\[
\langle \nabla \Phi_i(u) | f(t, u) \rangle \geq 0, \ \forall (t, u) \in [0, 1] \times \partial C, \ \forall i \in \alpha(u),
\]
or
\[
\langle \nabla \Phi_i(u) | f(t, u) \rangle \leq 0, \ \forall (t, u) \in [0, 1] \times \partial C, \ \forall i \in \alpha(u),
\]
where
\[
\alpha(u) := \{i \in \{1, \ldots, r\} : \Phi_i(u) = 0\}
\]
then
\[
x' = f(t, x), \ x(0) = x(1)
\]
has at least one solution taking values in \( \overline{C} \)

first existence thm : \( C = B_R, \ r = 1, \ \Phi_1(u) = \frac{1}{2}(|u|^2 - R^2) \)
Gustafson-Schmitt’s theorem

- $C$ open convex neighborhood of 0 in $\mathbb{R}^n$
- $\forall u \in \partial C$, $\exists \nu(u) \in \mathbb{R}^n \setminus \{0\} : \langle \nu(u) | u \rangle > 0$ and
  $C \subset \{ v \in \mathbb{R}^n : \langle \nu(u) | v - u \rangle < 0 \}$
- $\nu : \partial C \rightarrow \mathbb{R}^n \setminus \{0\} : \text{outer normal field on } \partial C$
Gustafson-Schmitt’s theorem

- \( C \) open convex neighborhood of 0 in \( \mathbb{R}^n \)
- \( \forall u \in \partial C, \exists \nu(u) \in \mathbb{R}^n \setminus \{0\} : \langle \nu(u)|u \rangle > 0 \) and \( C \subset \{ v \in \mathbb{R}^n : \langle \nu(u)|v - u \rangle < 0 \} \)
- \( \nu : \partial C \to \mathbb{R}^n \setminus \{0\} : \text{outer normal field on } \partial C \)

Gustafson-Schmitt’s thm. If \( \exists C, \) bounded convex open neighborhood of 0 in \( \mathbb{R}^n \), and \( \nu, \) outer normal field on \( \partial C \) : either

\[ \langle \nu(u)|f(t,u) \rangle > 0, \forall (t,u) \in [0,1] \times \partial C \]

or

\[ \langle \nu(u)|f(t,u) \rangle < 0, \forall (t,u) \in [0,1] \times \partial C, \]

then

\[ x' = f(t,x), \quad x(0) = x(1) \]

has at least one solution taking values in \( C \)

\[ \langle v(u), f(t,u) \rangle \geq 0 \]

\[ \langle v(u), f(t,u) \rangle \leq 0 \]
Comparison of the results

- KRASNOSELSKI’S monograph not quoted by GUSTAFSON-SCHMITT
- special case $C = B_R$ explicetly mentioned by GUSTAFSON-SCHMITT
Comparison of the results

- **Krasnosel’skii’s monograph not quoted by Gustafson-Schmitt**

- special case $C = B_R$ explicitly mentioned by **Gustafson-Schmitt**

- J.M., *Diford 74* (Stará Lesná 1974), I, 37–60:
  - connection Krasnosel’skii–Gustafson-Schmitt explicited
  - extension of Gustafson-Schmitt’s thm to weak inequalities

- Krasnosel’skii’s thm special case of extended Gustafson-Schmitt’s thm
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- several generalizations of Gustafson-Schmitt’s thm in
  - J.M., *Diford 74*
Nonlocal terminal BVP

- \( h : [0, 1] \rightarrow \mathbb{R} \) nondecreasing, \( \int_0^1 dh(s) = 1 \), \( h(0) < h(\alpha) \) for some \( \alpha \in (0, 1) \)

**Thm.** If \( \exists C \), open, bounded, convex neighborhood of 0 in \( \mathbb{R}^n \) and \( \nu \), outer normal field on \( \partial C \) :
\[
\langle \nu(u) | f(t, u) \rangle \geq 0, \quad \forall (t, u) \in [0, 1] \times \partial C,
\]
then
\[
x' = f(t, x), \quad x(1) = \int_0^1 dh(s)x(s)
\]
has at least one solution taking values in \( \overline{C} \)
Nonlocal terminal BVP

- $h : [0, 1] \rightarrow \mathbb{R}$ nondecreasing, $\int_0^1 dh(s) = 1$, $h(0) < h(\alpha)$ for some $\alpha \in (0, 1)$

- **Thm.** If $\exists C$, open, bounded, convex neighborhood of $0$ in $\mathbb{R}^n$ and $\nu$, outer normal field on $\partial C$:
  \[
  \langle \nu(u) | f(t, u) \rangle \geq 0, \quad \forall (t, u) \in [0, 1] \times \partial C,
  \]
  then
  \[
  x' = f(t, x), \quad x(1) = \int_0^1 dh(s)x(s)
  \]
  has at least one solution taking values in $\overline{C}$


- special case: multipoint boundary conditions
Nonlocal initial BVP

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**Thm.** If $\exists C$, open, bounded, convex neighborhood of 0 in $\mathbb{R}^n$ and $\nu$, outer normal field on $\partial C$:

$$\langle \nu(u) | f(t, u) \rangle \leq 0, \; \forall (t, u) \in [0, 1] \times \partial C,$$

then

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has at least one solution taking values in $\overline{C}$
Nonlocal initial BVP

- \( h : [0, 1] \to \mathbb{R} \) non decreasing, \( \int_{0}^{1} dh(s) = 1 \), \( h(\alpha) < h(1) \) for some \( \alpha \in (0, 1) \)

**Thm.** If \( \exists C \), open, bounded, convex neighborhood of \( 0 \) in \( \mathbb{R}^n \) and \( \nu \), outer normal field on \( \partial C \) :
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\langle \nu(u) | f(t, u) \rangle \leq 0, \quad \forall (t, u) \in [0, 1] \times \partial C,
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\[
x' = f(t, x), \quad x(0) = \int_{0}^{1} dh(s)x(s)
\]
has at least one solution taking values in \( \overline{C} \)


- special case: multipoint boundary conditions
Cor. If \( h : [0, 1] \to \mathbb{R} \) is nondecreasing, \( \int_0^1 dh(s) = 1 \), \( h(0) < h(\alpha) \) for some \( \alpha \in (0, 1) \), and if
\[
\exists R > 0 : \langle u | f(t, u) \rangle \geq 0, \quad \forall (t, u) \in [0, 1] \times \partial B_R,
\]
then
\[
x' = f(t, x), \quad x(1) = \int_0^1 dh(s)x(s)
\]
has at least one solution taking values in \( \overline{B}_R \).
Special case of a ball

**Cor.** If \( h : [0, 1] \to \mathbb{R} \) is nondecreasing, \( \int_0^1 dh(s) = 1 \), \( h(0) < h(\alpha) \) for some \( \alpha \in (0, 1) \), and if
\[
\exists R > 0 \ : \ \langle u|f(t,u) \rangle \geq 0, \ \forall (t,u) \in [0,1] \times \partial B_R,
\]
then
\[
x' = f(t,x), \ x(1) = \int_0^1 dh(s)x(s)
\]
has at least one solution taking values in \( \overline{B_R} \).

**Cor.** If \( h : [0, 1] \to \mathbb{R} \) is nondecreasing, \( \int_0^1 dh(s) = 1 \), \( h(\alpha) < h(1) \) for some \( \alpha \in (0, 1) \), and if
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x' = f(t,x), \ x(0) = \int_0^1 dh(s)x(s)
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has at least one solution taking values in \( \overline{B_R} \).
Remarks, questions and strategy

- in J.M.–S-D’s thms, the sense of inequality for \( \langle u | f(t, u) \rangle \) depends on the BC, in contrast with the periodic case.

- however \( \forall \lambda \in \mathbb{R} \setminus \{0\} \), \( x' = \lambda x + e(t), \)
  \( x(1) = \frac{1}{2}[x(1/2) + x(0)] \) or \( x(0) = \frac{1}{2}[x(1/2) + x(1)] \)
  has a solution \( \forall e \in C([0, 1], \mathbb{R}^n) \)

- can we have existence with the opposite sign?
Remarks, questions and strategy

- in J.M.–S-D’s thms, the sense of inequality for $\langle u|f(t, u) \rangle$ depends on the BC, in contrast with the periodic case

- however $\forall \lambda \in \mathbb{R} \setminus \{0\}$, $x' = \lambda x + e(t)$,
  
  $x(1) = \frac{1}{2}[x(1/2) + x(0)]$ or $x(0) = \frac{1}{2}[x(1/2) + x(1)]$

  *has a solution* $\forall e \in C([0, 1], \mathbb{R}^n)$

- can we have existence with the opposite sign?

- we construct counterexamples to show the answer is no

  - 2-dimensional eigenvalue problems $z' = \lambda z$ with 3-point BC

  - use Fredholm alternative to obtain forcing terms $e(t)$:
    
    $z' = \lambda z + e(t)$ + BC has no solution for $\lambda$ eigenvalue

  - show that this non-homogeneous problem written as a
    
    2-dimensional system $x' = f(t, x)$ satisfies the conditions of
    
    the J.M.–S-D’s corollaries with opposite signs for $\langle u|f(t, u) \rangle$
Eigenvalue problems

\( \lambda \in \mathbb{C}, \ z : [0, 1] \rightarrow \mathbb{C} \)

\( z' = \lambda z, \ z(1) = \frac{1}{2} [z(0) + z(1/2)] \)

- eigenvalues: \( 2k(2\pi i), \ -\log 4 + (2k + 1)(2\pi i) \ (k \in \mathbb{Z}) \)

- all contained in the left half plane and imaginary axis
Eigenvalue problems

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\[ z' = \lambda z, \ z(1) = \frac{1}{2}[z(0) + z(1/2)] \]
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*all complex, except 0*
Eigenvalue problems

\[ \lambda \in \mathbb{C}, \; z : [0, 1] \rightarrow \mathbb{C} \]
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- all contained in the right half plane and imaginary axis

*all complex, except 0*

**Rem.** \( z' = \lambda z, \; z(0) = z(1) \) has the eigenvalues
\( k(2\pi i), \; (k \in \mathbb{Z}) \)
Half of those eigenvalues move to \( \Re z = \log 4 \) (resp. \( \Re z = -\log 4 \)) for the three-point BC
$x(1) = (1/2)[x(0) + x(1/2)] \quad x(0) = x(1) \quad x(0) = (1/2)[x(1/2) + x(1)]$
Fredholm alternative

**prop.** \( \lambda \) is an eigenvalue of

\[
z' = \lambda z, \quad z(1) = \frac{1}{2}[z(0) + z(1/2)]
\]

(resp. \( z'(t) = \lambda z(t), \quad z(0) = \frac{1}{2}[z(1/2) + z(1)] \))

\(\Leftrightarrow\) \( \exists e_T \in C([0,1], \mathbb{C}) \) (resp. \( \exists e_I \in C([0,1], \mathbb{C}) \)) :

\[
z'(t) = \lambda z(t) + e_T(t), \quad z(1) = \frac{1}{2}[z(0) + z(1/2)]
\]

(resp. \( z'(t) = \lambda z(t) + e_I(t), \quad z(0) = \frac{1}{2}[z(1/2) + z(1)] \))

has no solution
Fredholm alternative

**prop.** $\lambda$ is an eigenvalue of

$$z' = \lambda z, \quad z(1) = \frac{1}{2}[z(0) + z(1/2)]$$

(resp. $z'(t) = \lambda z(t), \quad z(0) = \frac{1}{2}[z(1/2) + z(1)]$)

$\iff \exists e_T \in C([0, 1], \mathbb{C})$ (resp. $\exists e_I \in C([0, 1], \mathbb{C})$) :

$$z'(t) = \lambda z(t) + e_T(t), \quad z(1) = \frac{1}{2}[z(0) + z(1/2)]$$

(resp. $z'(t) = \lambda z(t) + e_I(t), \quad z(0) = \frac{1}{2}[z(1/2) + z(1)]$)

has no solution

**proof.** (1st case; 2nd analogous)

- $Lz := z' - z = e(t), \quad z(1) = \frac{1}{2}[z(0) + z(1/2)]$

  has a unique solution $z = L^{-1}e$

- $L^{-1} : C([0, 1], \mathbb{C}) \to C([0, 1], \mathbb{C})$ is compact

- EV problem $\iff z = (\lambda - 1)L^{-1}z + L^{-1}e$

- Riesz theory $\Rightarrow$ Fredholm alternative holds
Terminal type counterexample

\[ e_T \in C([0, 1], \mathbb{C}) : \]
\[ z'(t) = (-\log 4 + 2\pi i)z(t) + e_T(t), \quad z(1) = \frac{1}{2}[z(0) + z(1/2)] \]
has no solution
Terminal type counterexample

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\[ z'(t) = (-\log 4 + 2\pi i)z(t) + e_T(t), \quad z(1) = \frac{1}{2}[z(0) + z(1/2)] \]

has no solution

\[ z(t) := x_1(t) + ix_2(t), \quad e_T(t) = e_{T,1}(t) + ie_{T,2}(t) \]

\[ x'_1(t) = -(\log 4)x_1(t) - 2\pi x_2(t) + e_{T,1}(t) \]
\[ x'_2(t) = 2\pi x_1(t) - (\log 4)x_2(t) + h_{T,2}(t) \]
\[ x_1(1) = \frac{1}{2}[x_1(0) + x_1(1/2)] \]
\[ x_2(1) = \frac{1}{2}[x_2(0) + x_2(1/2)] \]
**Terminal type counterexample**

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\[
z'(t) = (-\log 4 + 2\pi i)z(t) + e_T(t), \quad z(1) = \frac{1}{2}[z(0) + z(1/2)]
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has no solution

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z(t) := x_1(t) + ix_2(t), \quad e_T(t) = e_{T,1}(t) + ie_{T,2}(t)
\]

\[
x_1'(t) = -(\log 4)x_1(t) - 2\pi x_2(t) + e_{T,1}(t)
\]
\[
x_2'(t) = 2\pi x_1(t) - (\log 4)x_2(t) + h_{T,2}(t)
\]

\[
x_1(1) = \frac{1}{2}[x_1(0) + x_1(1/2)]
\]
\[
x_2(1) = \frac{1}{2}[x_2(0) + x_2(1/2)]
\]

\[
f(t, u) := \)
\[
-(\log 4)u_1 - 2\pi u_2 + e_{T,1}(t), 2\pi u_1 - (\log 4)u_2 + e_{T,2}(t)
\]

\[
\langle u | f(t, u) \rangle = -(\log 4)(u_1^2 + u_2^2) + u_1 e_{T,1}(t) + u_2 e_{T,2}(t)
\]
\[
\leq -(\log 4)|u|^2 + |e_T(t)||u| < 0 \quad \text{when} \quad |u| \geq R \gg 0
\]
Initial type counterexample

\[ e_I \in C([0, 1], \mathbb{C}) : \]
\[ z'(t) = (\log 4 + 2\pi i)z(t) + e_I(t), \quad z(1) = \frac{1}{2}[z(0) + z(1/2)] \]

has no solution
Initial type counterexample

\[ e_I \in C([0, 1], \mathbb{C}) : \]
\[ z'(t) = (\log 4 + 2\pi i)z(t) + e_I(t), \quad z(1) = \frac{1}{2}[z(0) + z(1/2)] \]

has no solution

\[ z(t) := x_1(t) + ix_2(t), \quad e_I(t) = e_{I,1}(t) + ie_{I,2}(t) \]

\[ x_1'(t) = -(\log 4)x_1(t) - 2\pi x_2(t) + e_{I,1}(t) \]
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Initial type counterexample

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\[ z'(t) = (\log 4 + 2\pi i)z(t) + e_I(t), \quad z(1) = \frac{1}{2}[z(0) + z(1/2)] \]

has no solution

\[ z(t) := x_1(t) + ix_2(t), \quad e_I(t) = e_{I,1}(t) + ie_{I,2}(t) \]

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\[ x_2(1) = \frac{1}{2}[x_2(0) + x_2(1/2)] \]

\[ f(t, u) := \]
\[ ((\log 4)u_1 - 2\pi u_2 + e_{I,1}(t), 2\pi u_1 - (\log 4)u_2 + e_{I,2}(t)) \]

\[ \langle u | f(t, u) \rangle = (\log 4)(u_1^2 + u_2^2) + u_1 e_{I,1}(t) + u_2 e_{I,2}(t) \]
\[ \geq (\log 4)|u|^2 - |e_I||u| > 0 \quad \text{when} \quad |u| \geq R \gg 0 \]
Comments

- the symmetry-breaking with respect to reflection on imaginary axis for the spectra of the 3-point BVP explains the difference in existence conditions with respect to periodic conditions.

- despite of the same real spectrum \( \{0\} \) for the three problems, the presence of the complex spectrum in the left- or the right half plane influences like a ghost the existence conditions for solutions of the real nonlinear systems.
the symmetry-breaking with respect to reflection on imaginary axis for the spectra of the 3-point BVP explains the difference in existence conditions with respect to periodic conditions despite of the same real spectrum \{0\} for the three problems, the presence of the complex spectrum in the left- or the right half plane influences like a ghost the existence conditions for solutions of the real nonlinear systems maybe extra conditions upon \(f\) could provide existence results with the sign conditions of the counterexamples strictly speaking, our counterexamples do not cover the case of \(n = 1\) or even of \(n\) odd. For \(n = 3\), add the equations

\[
x'_3 = -(\log 4)x_3 + \frac{\log 4}{4}(x_1 + x_2), \quad x_3(1) = \frac{1}{2}[x_3(0) + x_3(1/2)]
\]
or

\[
x'_3 = (\log 4)x_3 + \frac{\log 4}{4}(x_1 + x_2), \quad x_3(0) = \frac{1}{2}[x_3(1/2) + x_3(1)]
\]
Sharpness of the periodic case

- $z' = 2\pi iz + e^{2\pi it}, \quad z(0) = z(1)$
  \[\Leftrightarrow (e^{-2\pi it}z)' = 1, \quad z(0) = z(1)\] has no solution

- $z = x_1 + ix_2 \Rightarrow x_1' = -2\pi x_2 + \cos(2\pi t)$
  $x_2' = 2\pi x_1 + \sin(2\pi t), \quad x_1(0) = x_1(1), \quad x_2(0) = x_2(1)$
  has no solution
Sharpness of the periodic case

\[ z' = 2\pi iz + e^{2\pi it}, \quad z(0) = z(1) \]
\[ \Leftrightarrow (e^{-2\pi it}z)' = 1, \quad z(0) = z(1) \] has no solution

\[ z = x_1 + ix_2 \Rightarrow x'_1 = -2\pi x_2 + \cos(2\pi t) \]
\[ x'_2 = 2\pi x_1 + \sin(2\pi t), \quad x_1(0) = x_1(1), \quad x_2(0) = x_2(1) \]
has no solution

\[ f_1(t, x_1, x_2) = -2\pi x_2 + \cos(2\pi t) \]
\[ f_2(t, x_1, x_2) = 2\pi x_1 + \cos(2\pi it) \]
\[ x = (x_1, x_2), \quad f(t, x) = (f_1(t, x_1, x_2), f_2(t, x_1, x_2)) \]

\[ \langle x, f(t, x) \rangle = \cos(2\pi t)x_1 + \sin(2\pi t)x_2 \]

\[ x = R[\cos(2\pi\theta), \sin(2\pi\theta)] \in \partial B_R : \]
\[ \langle x, f(t, x) \rangle = R \cos[2\pi(t - \theta)] \quad (t, \theta \in [0, 1]) \]
takes both positive and negative values
Nonlocal BVP for 2nd order systems - 1

- \( g : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) continuous, \( h : [0, 1] \rightarrow \mathbb{R} \) non decreasing, \( \int_0^1 dh = 1, \ h(\alpha) > h(0) \) for some \( \alpha \in (0, 1) \)

- **Thm.** If \( \exists C \), open, bounded, convex neighborhood of 0 in \( \mathbb{R}^n \) and \( \nu \), outer normal field on \( \partial C \) :
  \[
  \langle \nu(v) | g(t, u, v) \rangle \geq 0, \ \forall (t, u, v) \in [0, 1] \times \overline{C} \times \partial C,
  \]
  then
  \[
  x'' = g(t, x, x'), \ x(0) = 0, \ x'(1) = \int_0^1 dh(s)x'(s)
  \]
  has at least one solution with \( x \) and \( x' \) taking values in \( \overline{C} \)

- Systems of ordinary differential equations with nonlocal boundary conditions – p.23/??
Nonlocal BVP for 2nd order systems - 1

- $g : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ continuous, $h : [0, 1] \rightarrow \mathbb{R}$ non decreasing, $\int_0^1 dh = 1$, $h(\alpha) > h(0)$ for some $\alpha \in (0, 1)$

**Thm.** If $\exists C$, open, bounded, convex neighborhood of 0 in $\mathbb{R}^n$ and $\nu$, outer normal field on $\partial C$ :

\[ \langle \nu(v) | g(t, u, v) \rangle \geq 0, \forall (t, u, v) \in [0, 1] \times \overline{C} \times \partial C, \]

then

\[ x'' = g(t, x, x'), \ x(0) = 0, \ x'(1) = \int_0^1 dh(s)x'(s) \quad (*) \]

has at least one solution with $x$ and $x'$ taking values in $\overline{C}$

**Corr.** If $\exists R > 0$ :

\[ \langle \nu | g(t, u, v) \rangle \geq 0, \forall (t, u, v) \in [0, 1] \times \overline{B}_R \times \partial \overline{B}_R, \]

then

\[ (*) \text{ has at least one solution with } x \text{ and } x' \text{ taking values in } \overline{B}_R \]

similar thm and corollary, with the same sign for \( \langle v, g(t, u, v) \rangle \), for the boundary conditions\( x(0) = 0, \ x'(0) = \int_0^1 dh(s)x'(s) \) when \( h(\alpha) < h(1) \) for some \( \alpha \in (0, 1) \)

adaptations of 1st order counterexamples show that both existence conclusions need not be true when \( \langle v, g(t, u, v) \rangle \leq 0 \)
similar thm and corollary, with the same sign for \( \langle v, g(t, u, v) \rangle \), for the boundary conditions \( x(0) = 0, \ x'(0) = \int_0^1 dh(s)x'(s) \) when \( h(\alpha) < h(1) \) for some \( \alpha \in (0, 1) \)

adaptations of 1st order counterexamples show that both existence conclusions need not be true when \( \langle v, g(t, u, v) \rangle \leq 0 \)

Cor. If \( \exists R > 0 : \) either
\[
\langle v|g(t, u, v)\rangle \geq 0, \ \forall (t, u, v) \in [0, 1] \times \overline{B}_R \times \partial B_R
\]
or
\[
\langle v|g(t, u, v)\rangle \leq 0, \ \forall (t, u, v) \in [0, 1] \times \overline{B}_R \times \partial B_R,
\]
then \( x'' = g(t, x, x'), \ x(0) = 0, \ x'(0) = x'(1) \)
has at least one solution with \( x \) and \( x' \) taking values in \( \overline{B}_R \)

proof. a special case of both problems above

adaptation of 1st order counterexample show that the result is sharp
Thank you for your kind attention!