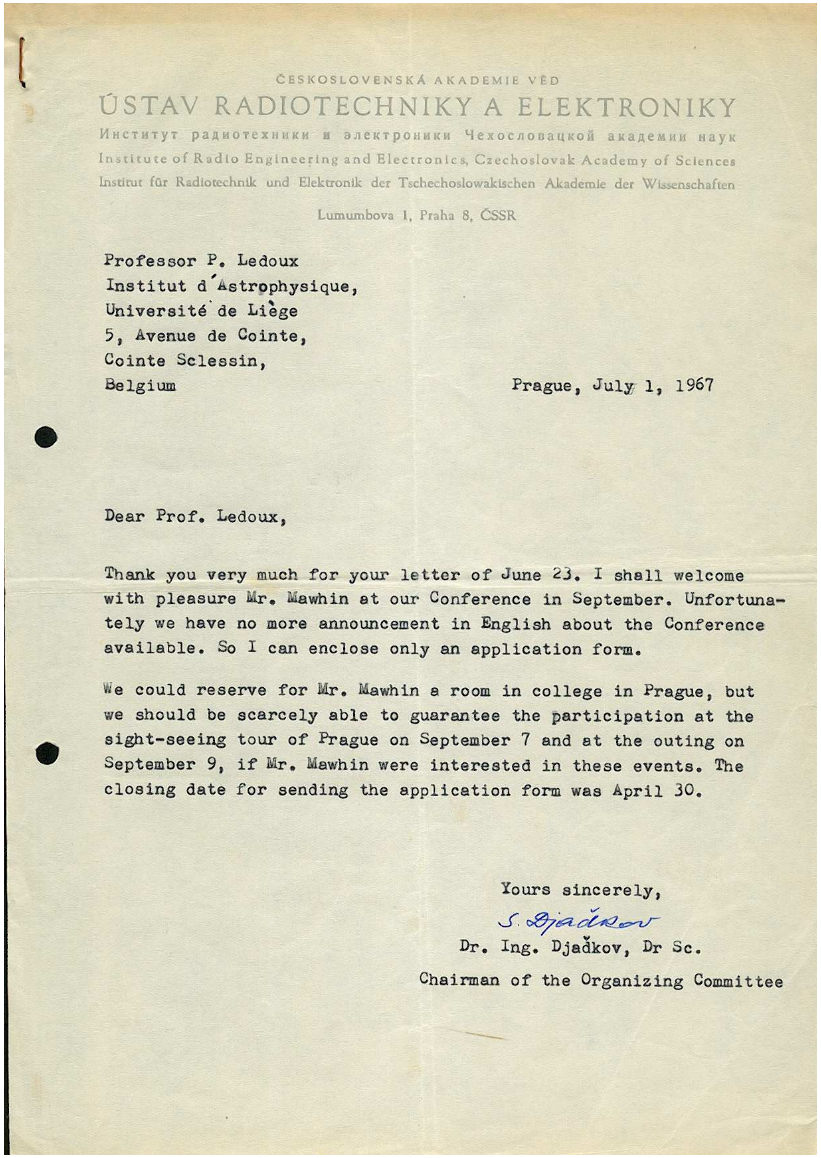
Systems of ordinary differential equations with nonlocal boundary conditions

Jean Mawhin

Université Catholique de Louvain

ICNO 4, Praha, Sept. 6-10, 1967



ČESKOSLOVENSKÁ AKADEMIE VĚD
ÚSTAV RADIOTECHNIKY A ELEKTRONIKY
Институт радиотехники и электроники Чехословацкой академии наук
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Lumumbova 1, Praha 8, ČSSR

Professor P. Ledoux
Institut d´Astrophysique,
Université de Liège
5, Avenue de Cointe,
Cointe Sclessin,
Belgium

Prague, July 1, 1967

Dear Prof. Ledoux,

Thank you very much for your letter of June 23. I shall welcome with pleasure Mr. Mawhin at our Conference in September. Unfortunately we have no more announcement in English about the Conference available. So I can enclose only an application form.

We could reserve for Mr. Mawhin a room in college in Prague, but we should be scarcely able to guarantee the participation at the sight-seeing tour of Prague on September 7 and at the outing on September 9, if Mr. Mawhin were interested in these events. The closing date for sending the application form was April 30.

Yours sincerely,
S. Djaďkov
Dr. Ing. Djaďkov, Dr Sc.
Chairman of the Organizing Committee

EQUADIFF III, Brno, 1972

CZECHOSLOVAK CONFERENCE ON DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS

EQUADIFF III

ORGANIZED BY CZECHOSLOVAK ACADEMY OF SCIENCES AND J. E. PURKYNĚ UNIVERSITY BRNO

BRNO • CZECHOSLOVAKIA

AUGUST 28 – SEPTEMBER 1 • 1972

MULDOON M., LORCH L., SZEGO P. Higher monotonicity properties of certain Sturm-Liouville functions

MULDOWNEY J. An intermediate value property for operators with applications to differential equations

PANTELEEV D., BAJNOV D. Ustojčivosť periodičeskikh rešenij kvazilinejnoj avtonomnoj sistemy s zapazdyvaniem v slučae trekhkratnykh kornej amplitudnykh uravnenij

RAHMI A. K. The resonance case in linear differential equations

REICH L. Über die Abschätzung des Wachstumsordnung in der Fuchsschen Theorie

SCHMITT B. An index useful for the research of periodic solutions of periodic second-order differential equations

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ANGER G. Inverse Probleme des Potentialtheorie (Geophysik)

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KAČUR J. Method of Rothe and nonlinear parabolic boundary value problems of arbitrary order

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Thank you to my Czech colleagues for 50 years of fruitful collaboration and frienship

and thank you to my friends of Brno for 45 years of warm hospitality

A classical existence theorem

- $\mathbb{R}^n,\ \langle\cdot|\cdot\rangle,\ |\cdot|,\ B_R;\ \ f \in C([0,1]\times\mathbb{R}^n, \mathbb{R}^n)$
- **Thm.** *If* $\exists\, R > 0:$

  either

  $$\langle u|f(t,u)\rangle \geq 0,\ \forall\,(t,u) \in [0,1]\times\partial B_R,$$

  or

  $$\langle u|f(t,u)\rangle \leq 0,\ \forall\,(t,u) \in [0,1]\times\partial B_R,$$

  then

  $$x' = f(t,x),\ x(0) = x(1)$$

  has at least one solution taking values in $\overline{B}_R$

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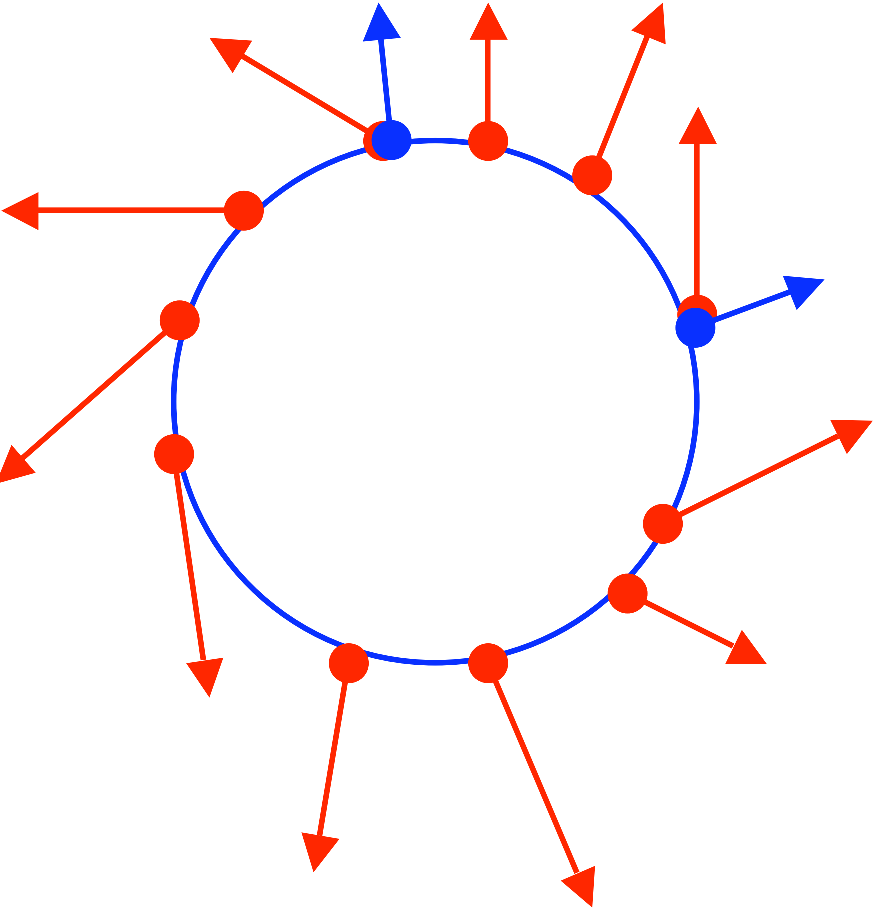
  then

  $x' = f(t,x), \ x(0) = x(1)$

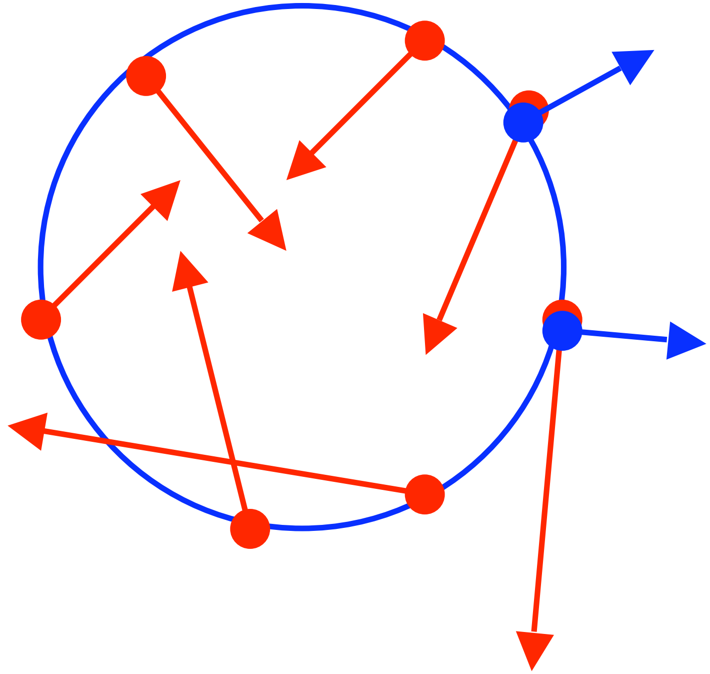
  has at least one solution taking values in $\overline{B}_R$
- equivalent statements : $\tau = 1 - t$
- nonlinear version of the linear result :

  Prop. $\forall\,\lambda \in \mathbb{R}\setminus\{0\}, \ \forall\, e \in C([0,T],\mathbb{R}^n)$

  $x' = \lambda x + e(t), \ x(0) = x(1)$ *has a solution*



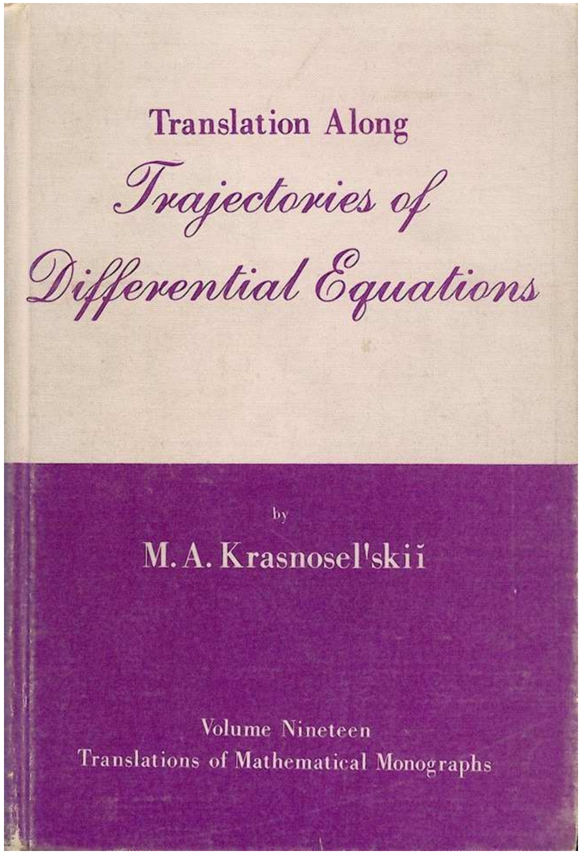
$$<u,f(t,u)> \geq 0$$



$$<u,f(t,u)> \leq 0$$

References

- special case (not mentioned !) of Theorem 3.2 in M.A. KRASNOSEL'SKII, *The Operator of Translation along the Trajectories of Differential Equations*, Moscow, 1966



Krasnosel'skii's theorem

- **Thm.** *If* $\exists\, C$, *bounded open convex set,* $\Phi_i \in C^1(\mathbb{R}^n, \mathbb{R})$ $(i = 1, \dots, r)$: $\overline{C} = \{u \in \mathbb{R}^n : \Phi_i(u) \le 0 \ \ (i = 1, \dots, r)\}$, $\Phi_i(u) = 0$ *for some* $u \in \partial C \ \Rightarrow\ \nabla\Phi_i(u) \neq 0$, *and*
  either
  $$\langle \nabla\Phi_i(u) | f(t, u)\rangle \ge 0,\ \forall\ (t, u) \in [0, 1] \times \partial C,\ \forall\, i \in \alpha(u),$$
  or
  $$\langle \nabla\Phi_i(u) | f(t, u)\rangle \le 0,\ \forall\ (t, u) \in [0, 1] \times \partial C,\ \forall\, i \in \alpha(u),$$
  where
  $$\alpha(u) := \{i \in \{1, \dots, r\} : \Phi_i(u) = 0\}$$
  then
  $$x' = f(t, x),\ x(0) = x(1)$$
  has at least one solution taking values in $\overline{C}$
- first existence thm : $C = B_R,\ r = 1,\ \Phi_1(u) = \frac{1}{2}(|u|^2 - R^2)$

Gustafson-Schmitt's theorem

- C open convex neighborhood of 0 in $\mathbb{R}^n$
  - $\forall\, u \in \partial C,\ \exists\, \nu(u) \in \mathbb{R}^n \setminus \{0\}\ :\ \langle \nu(u) | u \rangle > 0$ and $C \subset \{v \in \mathbb{R}^n : \langle \nu(u) | v - u \rangle < 0\}$
  - $\nu : \partial C \to \mathbb{R}^n \setminus \{0\}$: **outer normal field** on ∂C

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  - $\nu : \partial C \to \mathbb{R}^n \setminus \{0\}$: **outer normal field** on ∂C
- **Gustafson-Schmitt's thm.** *If $\exists\, C$, bounded convex open neighborhood of 0 in $\mathbb{R}^n$, and ν, outer normal field on ∂C : either*

  $$\langle \nu(u) | f(t,u) > 0,\ \forall\, (t,u) \in [0,1] \times \partial C$$

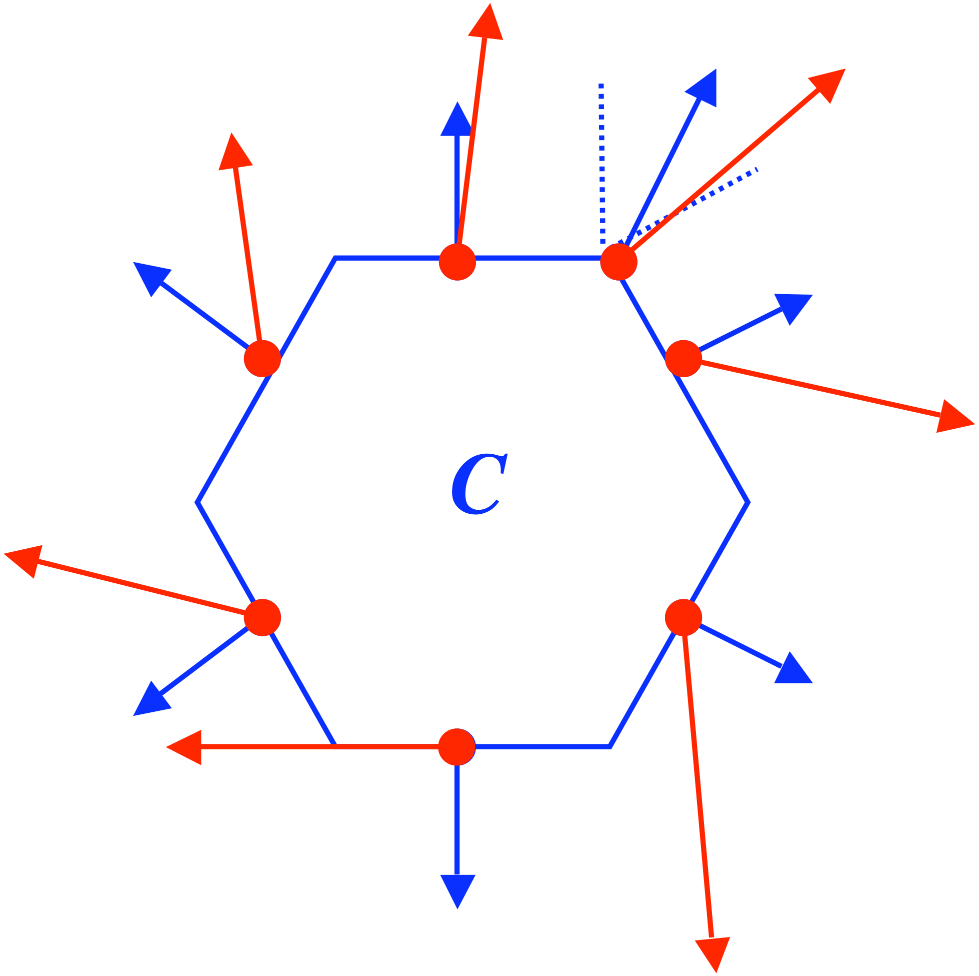
  or

  $$\langle \nu(u) | f(t,u) < 0,\ \forall\, (t,u) \in [0,1] \times \partial C,$$

  then

  $$x' = f(t,x),\ x(0) = x(1)$$

  has at least one solution taking values in C
- *Proc. Amer. Math. Soc.* **42** (1974), 161–166



$<\nu(u), f(t,u)> \geq 0$ $\quad$ $<\nu(u), f(t,u)> \leq 0$

Comparison of the results

- KRASNOSEL'SKII'S monograph not quoted by GUSTAFSON-SCHMITT
- special case $C = B_R$ explicitely mentioned by GUSTAFSON-SCHMITT

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- J.M., *Diford 74* (Stará Lesná 1974), I, 37–60 :
  - connection Krasnosel'skii–Gustafson-Schmitt explicited
  - extension of Gustafson-Schmitt's thm to weak inequalities
- Krasnosel'skii's thm special case of extended Gustafson-Schmitt's thm

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- Krasnosel'skii's thm special case of extended Gustafson-Schmitt's thm
- several generalizations of Gustafson-Schmitt's thm in
  - J.M., *Diford 74*
  - GAINES-J.M., *Coincidence Degree and Nonlinear Differential Equations*, Springer, 1977

Nonlocal terminal BVP

- $h : [0,1] \to \mathbb{R}$ nondecreasing, $\int_0^1 dh(s) = 1$, $h(0) < h(\alpha)$ for some $\alpha \in (0,1)$
- **Thm.** *If* $\exists\, C$, *open, bounded, convex neighborhood of* 0 *in* $\mathbb{R}^n$ *and* ν, *outer normal field on* ∂C *:*
  $\langle \nu(u) | f(t,u) \rangle \geq 0,\ \forall\, (t,u) \in [0,1] \times \partial C,$
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- J.M.-K. SZYMAŃSKA-DĘBOWSKA, *J. Nonlin. Convex Anal.* **18** (2017), 149–160 (more general versions are given there)
- special case : multipoint boundary conditions

Nonlocal initial BVP

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Special case of a ball

- **Cor.** *If* $h : [0,1] \to \mathbb{R}$ *is nondecreasing,* $\int_0^1 dh(s) = 1$, $h(0) < h(\alpha)$ *for some* $\alpha \in (0,1)$, *and if*
  $\exists\, R > 0 \;:\; \langle u | f(t,u) \rangle \geq 0,\ \forall\, (t,u) \in [0,1] \times \partial B_R$,
  then
  $x' = f(t,x),\ x(1) = \int_0^1 dh(s) x(s)$
  has at least one solution taking values in $\overline{B}_R$.

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  then
  $x' = f(t,x),\ x(0) = \int_0^1 dh(s) x(s)$
  has at least one solution taking values in $\overline{B}_R$

Remarks, questions and strategy

- in J.M.–S-D's thms, the sense of inequality for $\langle u | f(t,u) \rangle$ depends on the BC, in contrast with the periodic case
- however $\forall\, \lambda \in \mathbb{R} \setminus \{0\}$, $x' = \lambda x + e(t)$,
  $x(1) = \frac{1}{2}[x(1/2) + x(0)]$ *or* $x(0) = \frac{1}{2}[x(1/2) + x(1)]$
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- can we have existence with the opposite sign ?
- we construct counterexamples to show the answer is **no**
  - 2-dimensional eigenvalue problems $z' = \lambda z$ with 3-point BC
  - use Fredholm alternative to obtain forcing terms $e(t)$:
    $z' = \lambda z + e(t)$ + BC has no solution for λ eigenvalue
  - show that this non-homogeneous problem written as a 2-dimensional system $x' = f(t,x)$ satisfies the conditions of the J.M.–S-D's corollaries with opposite signs for $\langle u | f(t,u) \rangle$

Eigenvalue problems

- $\lambda \in \mathbb{C},\ z : [0,1] \to \mathbb{C}$
- $z' = \lambda z,\ z(1) = \frac{1}{2}[z(0) + z(1/2)]$
  - eigenvalues : $2k(2\pi i),\ -\log 4 + (2k+1)(2\pi i)\ \ (k \in \mathbb{Z})$
  - all contained in the left half plane and imaginary axis

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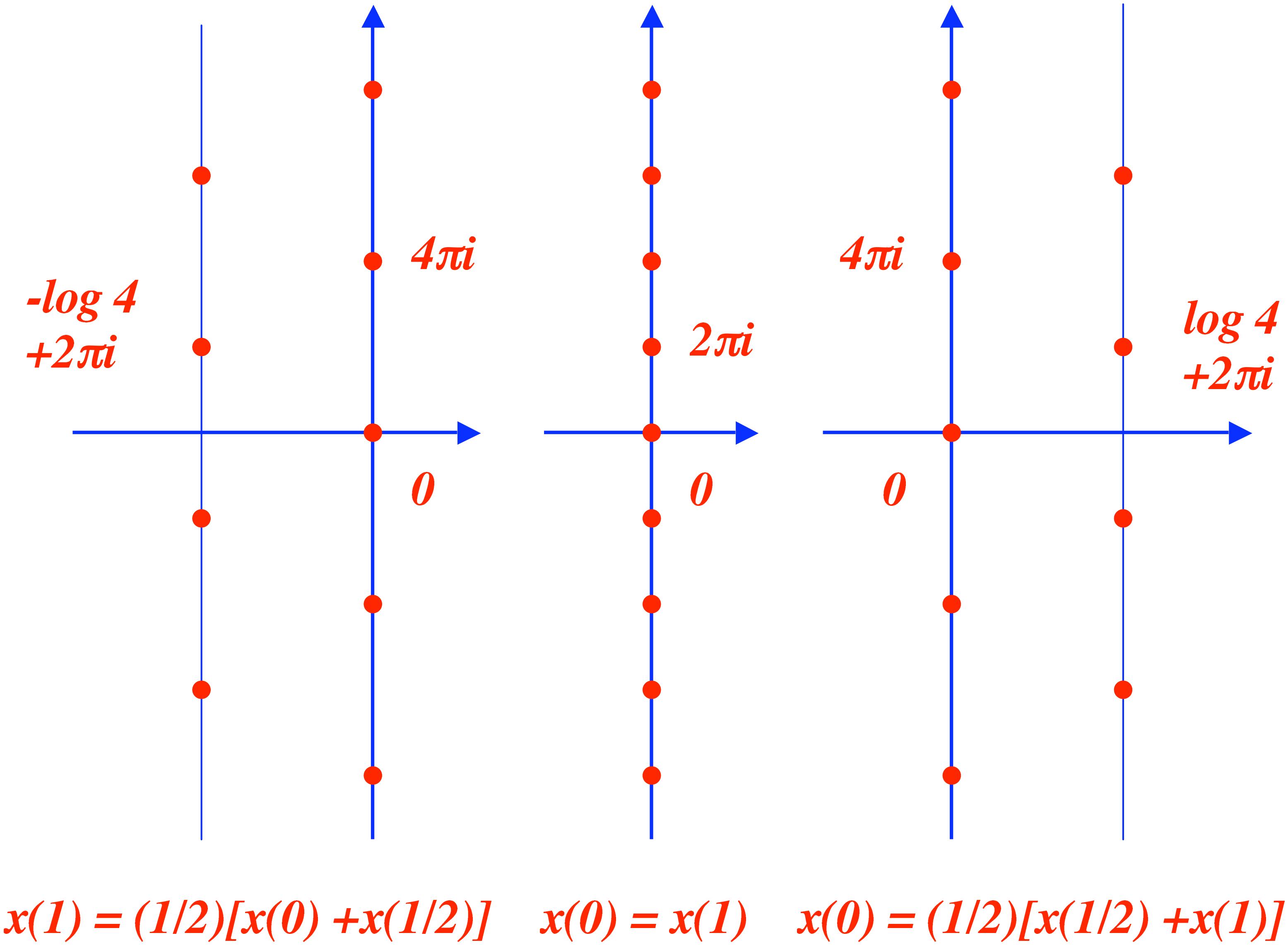
  - all contained in the right half plane and imaginary axis
- *all complex, except* 0
- **Rem.** $z' = \lambda z,\ z(0) = z(1)$ has the eigenvalues $k(2\pi i),\ (k \in \mathbb{Z})$
  Half of those eigenvalues move to $\Re z = \log 4$ (resp. $\Re z = -\log 4$) for the three-point BC


-log 4
+2πi
4πi
0
2πi
0
4πi
0
log 4
+2πi
x(1) = (1/2)[x(0) +x(1/2)]
x(0) = x(1)
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Fredholm alternative

- **prop.** λ *is an eigenvalue of*

  $z' = \lambda z,\ z(1) = \frac{1}{2}[z(0) + z(1/2)]$

  (resp. $z'(t) = \lambda z(t),\ z(0) = \frac{1}{2}[z(1/2) + z(1)]$*)*

  $\Leftrightarrow\ \exists\, e_T \in C([0,1],\mathbb{C})$ *(resp.* $\exists\, e_I \in C([0,1],\mathbb{C})$*) :*

  $z'(t) = \lambda z(t) + e_T(t),\ z(1) = \frac{1}{2}[z(0) + z(1/2)]$

  (resp. $z'(t) = \lambda z(t) + e_I(t),\ z(0) = \frac{1}{2}[z(1/2) + z(1)]$*)*

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  has no solution

- **proof.** (1st case; 2nd analogous)
  - $Lz := z' - z = e(t),\ z(1) = \frac{1}{2}[z(0) + z(1/2)]$

    has a unique solution $z = L^{-1}e$
  - $L^{-1} : C([0,1],\mathbb{C}) \to C([0,1],\mathbb{C}$ is compact
  - EV problem $\Leftrightarrow$ $z = (\lambda - 1)L^{-1}z + L^{-1}e$
  - Riesz theory $\Rightarrow$ Fredholm alternative holds

Terminal type counterexample

- $e_T \in C([0,1],\mathbb{C})$:
  $z'(t) = (-\log 4 + 2\pi i)z(t) + e_T(t), \; z(1) = \frac{1}{2}[z(0) + z(1/2)]$
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  has no solution
- $z(t) := x_1(t) + ix_2(t),\ e_T(t) = e_{T,1}(t) + ie_{T,2}(t)$
- $x_1'(t) = -(\log 4)x_1(t) - 2\pi x_2(t) + e_{T,1}(t)$
  $x_2'(t) = 2\pi x_1(t) - (\log 4)x_2(t) + h_{T,2}(t)$
  $x_1(1) = \frac{1}{2}[x_1(0) + x_1(1/2)]$
  $x_2(1) = \frac{1}{2}[x_2(0) + x_2(1/2)]$

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  $x_2(1) = \frac{1}{2}[x_2(0) + x_2(1/2)]$
- $f(t,u) :=$
  $\big(-(\log 4)u_1 - 2\pi u_2 + e_{T,1}(t), 2\pi u_1 - (\log 4)u_2 + e_{T,2}(t)\big)$
- $\langle u | f(t,u)\rangle = -(\log 4)(u_1^2 + u_2^2) + u_1 e_{T,1}(t) + u_2 e_{T,2}(t)$
  $\leq -(\log 4)|u|^2 + |e_T(t)||u| < 0$ when $|u| \geq R \gg 0$

Initial type counterexample

- $e_I \in C([0,1],\mathbb{C})$:
  $z'(t) = (\log 4 + 2\pi i)z(t) + e_I(t),\ z(1) = \frac{1}{2}[z(0) + z(1/2)]$
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  $\geq (\log 4)|u|^2 - |e_I||u| > 0$ when $|u| \geq R \gg 0$

Comments

- the symmetry-breaking with respect to reflection on imaginary axis for the spectra of the 3-point BVP explains the difference in existence conditions with respect to periodic conditions
- despite of the same **real** spectrum $\{0\}$ for the three problems, the presence of the **complex** spectrum in the left- or the right half plane influences like a ghost the existence conditions for solutions of the real nonlinear systems

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- the symmetry-breaking with respect to reflection on imaginary axis for the spectra of the 3-point BVP explains the difference in existence conditions with respect to periodic conditions
- despite of the same **real** spectrum $\{0\}$ for the three problems, the presence of the **complex** spectrum in the left- or the right half plane influences like a ghost the existence conditions for solutions of the real nonlinear systems
- maybe **extra conditions** upon f could provide existence results with the sign conditions of the counterexamples
- strictly speaking, our counterexamples do not cover the case of $n = 1$ or even of n odd. For $n = 3,$ add the equations
$$x_3' = -(\log 4)x_3 + \frac{\log 4}{4}(x_1 + x_2),\ x_3(1) = \frac{1}{2}[x_3(0) + x_3(1/2)]$$
or
$$x_3' = (\log 4)x_3 + \frac{\log 4}{4}(x_1 + x_2),\ x_3(0) = \frac{1}{2}[x_3(1/2) + x_3(1)]$$

Sharpness of the periodic case

- $z' = 2\pi i z + e^{2\pi i t},\ z(0) = z(1)$
  $\Leftrightarrow\ (e^{-2\pi i t} z)' = 1,\ z(0) = z(1)$ has no solution
- $z = x_1 + i x_2\ \Rightarrow\ x_1' = -2\pi x_2 + \cos(2\pi t)$
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- $f_1(t, x_1, x_2) = -2\pi x_2 + \cos(2\pi t)$
  $f_2(t, x_1, x_2) = 2\pi x_1 + \cos(2\pi i t)$
  $x = (x_1, x_2),\ f(t, x) = (f_1(t, x_1, x_2), f_2(t, x_1, x_2))$
- $\langle x, f(t, x)\rangle = \cos(2\pi t) x_1 + \sin(2\pi t) x_2$
- $x = R[\cos(2\pi\theta), \sin(2\pi\theta)] \in \partial B_R$:
  $\langle x, f(t, x)\rangle = R\cos[2\pi(t - \theta)]\ \ (t, \theta \in [0, 1])$
  takes both positive and negative values

Nonlocal BVP for 2nd order systems - 1

- $g : [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ continuous, $h : [0,1] \to \mathbb{R}$ non decreasing, $\int_0^1 dh = 1$, $h(\alpha) > h(0)$ for some $\alpha \in (0,1)$
- **Thm.** *If* $\exists\, C$ *, open, bounded, convex neighborhood of* 0 *in* $\mathbb{R}^n$ *and* ν, *outer normal field on* ∂C *:*
  $\langle \nu(v) | g(t,u,v) \rangle \geq 0, \ \forall\, (t,u,v) \in [0,1] \times \overline{C} \times \partial C$,
  then
  $$x'' = g(t,x,x'), \ x(0) = 0, \ x'(1) = \int_0^1 dh(s) x'(s) \qquad (*)$$
  has at least one solution with x *and* x' *taking values in* $\overline{C}$

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- **Corr.** *If* $\exists\, R > 0$ *:*
  $\langle v | g(t,u,v) \rangle \geq 0, \ \forall\, (t,u,v) \in [0,1] \times \overline{B}_R \times \partial B_R$, *then*
  (*) has at least one solution with x *and* x' *taking values in* $\overline{B}_R$
- J.M.-SZYMAŃSKA-DĘBOWSKA, *Proc. AMS* **145** (2017), 2023–2032

Nonlocal BVP for 2nd order systems - 2

- similar thm and corollary, with the same sign for $\langle v, g(t,u,v)\rangle$,
  for the boundary conditions $x(0) = 0,\ x'(0) = \int_0^1 dh(s)x'(s)$
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- adaptations of 1st order counterexamples show that both existence conclusions need not be true when $\langle v, g(t,u,v)\rangle \leq 0$

Nonlocal BVP for 2nd order systems - 2

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- adaptations of 1st order counterexamples show that both existence conclusions need not be true when $\langle v, g(t,u,v)\rangle \leq 0$
- **Cor.** *If* $\exists\, R > 0 \ :$ *either*
  $\langle v|g(t,u,v)\rangle \geq 0,\ \forall\,(t,u,v) \in [0,1] \times \overline{B}_R \times \partial B_R$
  or $\langle v|g(t,u,v)\rangle \leq 0,\ \forall\,(t,u,v) \in [0,1] \times \overline{B}_R \times \partial B_R$,
  then $x'' = g(t,x,x'),\ x(0) = 0,\ x'(0) = x'(1)$
  has at least one solution with x *and* x' *taking values in* $\overline{B}_R$
  - **proof.** a special case of both problems above
- adaptation of 1st order counterexample show that the result is sharp

Thank you for your kind attention !