Non-monotone traveling waves solutions for a monostable reaction-diffusion equations with delay

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Introduction

\[ u_t(t, x) = \Delta u(t, x) - u(t, x) + g(u(t - h, x)); \quad (1) \]

- \( h \geq 0 \) is the delay,
- \( g : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is the birth function.
Introduction

Delayed Diffusive Nicholson’s Blowflies Equation [Gurney, Blythe, Nisbet, 1980]

\[
    u_t(t, x) = \Delta u(t, x) - \delta u(t, x) + pu(t - h, x)e^{-u(t-h,x)} \tag{2}
\]
Introduction

Delayed Diffusive Nicholson’s Blowflies Equation [Gurney, Blythe, Nisbet, 1980]

\[ u_t(t, x) = \Delta u(t, x) - \delta u(t, x) + pu(t - h, x)e^{-u(t-h,x)} \]  \quad (2)

Diffusive Mackey-Glass equation [blood cell production model, 1977]

\[ u_t(t, x) = \Delta u(t, x) - u(t, x) + p \frac{u(t - h, x)}{1 + (u(t - h, x))^n}; \]  \quad (3)
Introduction

- \( g(0) = 0, \ g(\kappa) = \kappa > 0 \) is a \( C^2(\mathbb{R}_+) \) function.
- \( g'(0) > 1, \ g'\left(\kappa\right) < 1 \)
- If \( g \) is not strictly increasing between 0 and \( \kappa \), then it has a unique global maximum in \( x_M \)
- \( u_0 \equiv 0, \ u_\kappa \equiv \kappa \) are constant solutions to (1)
Introduction
\( u(t, x) \) is a traveling wave solution for (1) if it is positive and \( u(t, x) = \phi(x + ct) \), \( \phi : \mathbb{R} \rightarrow \mathbb{R} \) is a \( C^2(\mathbb{R}) \), \( \phi(-\infty) = 0 \), \( \phi(+\infty) = \kappa \).

\( c \) is the wave’s speed propagation.
\[ \phi''(t) - c\phi'(t) - \phi(t) + g(\phi(t - ch)) = 0, \tag{4} \]
\[
\phi(-\infty) = 0, \quad \phi(+\infty) = \kappa
\]
Implicit equation of $c^*(h, g'(\kappa))$:

$$\frac{\beta - \alpha}{\beta e^{-\alpha ch} - \alpha e^{-\beta ch}} = \frac{q^2 + q}{q^2 + 1},$$

$\alpha < 0 < \beta$ roots of $z^2 - cz - 1 = 0$, $q := g'(\kappa)$. 
Linearized equation about

- 0 equilibrium: \( \phi''(t) - c\phi'(t) - \phi(t) + g'(0)\phi(t - ch) = 0, \)
- \( \kappa \) equilibrium: \( \phi''(t) - c\phi'(t) - \phi(t) + g'(\kappa)\phi(t - ch) = 0, \)

Characteristic equations:

- 0 equilibrium: \( \chi_0(z) := z^2 - cz - 1 + g'(0)e^{-chz} = 0, \)
- \( \kappa \) equilibrium: \( \chi_\kappa(z) := z^2 - cz - 1 + g'(\kappa)e^{-chz} = 0, \)
Lemma

(a) There exist \( c_0 = c_0(h, g'(0)) > 0 \) such that the characteristic equation \( \chi_0(z) = 0 \) has exactly two simple real roots \( 0 < \lambda = \lambda(c) < \mu = \mu(c) \) if and only if \( c > c_0(h) \). Next, if \( c > c_0 \), then all complex roots \( \{\lambda_j\}_{j \geq 1} \) of this equation are simple and can be ordered in such a way that

\[
\ldots \leq R(\lambda_3) \leq R(\lambda_4) \leq R(\lambda_2) = R(\lambda_1) < \lambda < \mu. \tag{5}
\]

Finally \( c_0(h) \) is a decreasing function, with \( c_0(+\infty) = 0 \).

(b) Let \( q := g'(\kappa) \). There exist \( c_\kappa = c_\kappa(h) \in (0, +\infty] \) such that the characteristic equation \( \chi_\kappa(z) = 0 \) has three real roots \( \lambda_1 \leq \lambda_2 < 0 < \lambda_3 \) if and only if \( c \leq c_\kappa(h) \). Furthermore, \( c_\kappa(0) = +\infty \) and \( c_\kappa(h) \) is strictly decreasing in its domain, with \( c_\kappa(+\infty) = 0 \).
Theorem

If \( g(x) \leq g'(0)x \) and \( g(x) \leq g'(\kappa)(x-k) + k \), equation (4) has a monotone solution \( \phi(t) \) if and only if

\[
(h, c) \in D_L := \{(h, c) \mid c_0(h) \leq c \leq c_\kappa(h)\}
\]
Lemma

\[ ch = 1 \quad c_k(h) \quad c^*(h) \]

\[ h_1 \quad h \quad h_2 \]
Implicit equations of $c_\kappa(h)$:

$$
2 + \frac{\sqrt{c^4 h^2 + 4c^2 h^2 + 4}}{c^2 h^2 |q|} = \exp\left(1 + \frac{\sqrt{c^4 h^2 + 4c^2 h^2 + 4 - c^2 h}}{2}\right).
$$
Theorem (1)

If \( g \) is not strictly increasing in \([0, \kappa]\), then equation 1 has a slowly oscillating traveling wave for each \( (h, c) \in \mathcal{D}^* \setminus \mathcal{D}_L \)
Nicholson’s Equation

\[ u_t(t, x) = \Delta u(t, x) - \delta u(t, x) + pu(t - h, x)e^{-u(t-h,x)} \]  \hspace{1cm} (6)

- If \( \frac{p}{\delta} > 1 \) is a monostable reaction diffusion equation, with equilibria \( u_0 = 0 \) and \( u_\kappa = \ln(\frac{p}{\delta}) \)
Nicholson’s Equation

\[ u_t(t, x) = \Delta u(t, x) - \delta u(t, x) + pu(t - h, x)e^{-u(t-h,x)} \]  \hspace{1cm} (6)

- If \( p/\delta > 1 \) is a monostable reaction diffusion equation, with equilibria \( u_0 = 0 \) and \( u_\kappa = \ln(p/\delta) \)
- If \( 1 < p/\delta \leq e \) there exist a unique traveling wave solution and it must be monotone (showed using super- and sub-solution method for all \( c > c_0(h) \) in [So-Zou, 2001]).
- If \( e < p/\delta \leq e^2 \) then there exist traveling waves solutions for all \((h,c) \in D\) and there are monotone or slowly oscillating [E. Trofimchuk, V. Tkachenko and S. Trofimchuk, 2008], also there are monotone if \((h,c) \in D_L \subset D\) [A. Gomez, S. Trofimchuk, 2014] and slowly oscillating if \((h,c) \notin D\) (corollary of Theorem 1).
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- If \( p/\delta > 1 \) is a monostable reaction diffusion equation, with equilibria \( u_0 = 0 \) and \( u_\kappa = \ln(p/\delta) \)
- If \( 1 < p/\delta \leq e \) there exist a unique traveling wave solution and it must be monotone (showed using super- and sub-solution method for all \( c > c_0(h) \) in [So-Zou, 2001]).
- If \( e < p/\delta \leq e^2 \) then there exist traveling waves solutions for all \( (h, c) \in D^* \) and there are monotone or slowly oscillating [E. Trofimchuk, V. Tkachenko and S. Trofimchuk, 2008], also there are monotone if \( (h, c) \in D_L \subset D^* \) [A. Gomez, S. Trofimchuk, 2014] and slowly oscillating if \( (h, c) \in D^* \backslash D_L \) (corollary of Theorem 1)
Theorem
If \( \frac{p}{\delta} \in (e, e^2] \) the region \( \mathcal{D}_L \) can have one of the following geometric forms with \( \nu_0 \approx 2.808.. \) and \( h_a \) defined by
\[
\delta h_a e^{\delta h_a} = [e \ln(p/e\delta)]^{-1}
\]
Figure: $p/\delta > \nu_0$

there exist a maximal delay $h_0$ to the monotonicity
At minimum speed of propagation, the traveling waves solutions are monotone
A. Gomez and S. Trofimchuk, Global Continuation of monotone wavefronts, J. London Math Soc. (2) 89 (2014) 47-68.

References
