On positive solutions for (p, q)-Laplace equations with two parameters

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Based on the joint work with Mieko Tanaka
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Introduction

We consider the following (p,q)-Laplacian problem

$$\begin{cases} -\Delta_p u - \Delta_q u = \alpha |u|^{p-2}u + \beta |u|^{q-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (\mathcal{D}_{\alpha,\beta})$$

where $p > q > 1$, $\Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, and $\alpha, \beta \in \mathbb{R}$ are parameters; $\Omega \subset \mathbb{R}^N$ is a bounded domain, $N \geqslant 1$.

We are interested in the existence and multiplicity of positive solutions to $(\mathcal{D}_{\alpha,\beta})$ with respect to α and β.

Problem $(\mathcal{D}_{\alpha,\beta})$ corresponds to the C^1 energy functional $E_{\alpha,\beta} : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined as

$$E_{\alpha,\beta}(u) = \frac{1}{p}\left(\int_\Omega |\nabla u|^p \, dx - \alpha \int_\Omega |u|^p \, dx\right) + \frac{1}{q}\left(\int_\Omega |\nabla u|^q \, dx - \beta \int_\Omega |u|^q \, dx\right).$$

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$$\lambda_1(r) := \inf\left\{ \frac{\|\nabla u\|_r^r}{\|u\|_r^r} : \ u \in W_0^{1,r}(\Omega) \setminus \{0\} \right\}, \qquad r = p, q.$$

The corresponding (first) eigenfunction is denoted as φ_r.

Lemma

Let $u \in W_0^{1,r}(\Omega)$.

- *If $\gamma \leqslant \lambda_1(r)$, then*

$$\int_\Omega |\nabla u|^r \, dx - \gamma \int_\Omega |u|^r \, dx \geqslant 0.$$

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Brief historical remarks

• Neumann boundary conditions (either $\alpha = 0$ or $\beta = 0$) - [Mihăilescu, 2011], [Mihăilescu, Moroșanu, 2015], etc.

• Dirichlet boundary conditions - see the survey [Marano, Mosconi, 2017].

In [Tanaka, 2014] the following two problems were considered:

1) $$-\Delta_p u - \Delta_q u = \alpha |u|^{p-2} u \ \text{ in } \ \Omega, \quad u = 0 \ \text{ on } \ \partial\Omega,$$

for which it was proved

- the existence of a positive solution for $\alpha > \lambda_1(p)$;
- the nonexistence of solutions for $\alpha \leqslant \lambda_1(p)$,

and

2) $$-\Delta_p u - \Delta_q u = \beta |u|^{q-2} u \ \text{ in } \ \Omega, \quad u = 0 \ \text{ on } \ \partial\Omega,$$

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In [Motreanu, Tanaka, 2016] the following problem was considered:

$$-\Delta_p u - \Delta_q u = \lambda(|u|^{p-2}u + |u|^{q-2}u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \qquad (3)$$

and it was proved

- the existence of a positive solution when

$$\min\{\lambda_1(q), \lambda_1(p)\} < \lambda < \max\{\lambda_1(q), \lambda_1(p)\};$$

- the nonexistence of solutions when $\lambda \leqslant \min\{\lambda_1(q), \lambda_1(p)\}$.

- If $\lambda_1(q) < \lambda_1(p)$, then $E_{\lambda,\lambda}$ has a global minimum;
- If $\lambda_1(q) > \lambda_1(p)$, then $E_{\lambda,\lambda}$ has the mountain pass geometry.

In [Kajikiya, Tanaka, Tanaka, 2017], problem (3) was studied in more details in 1D case by means of the time map.

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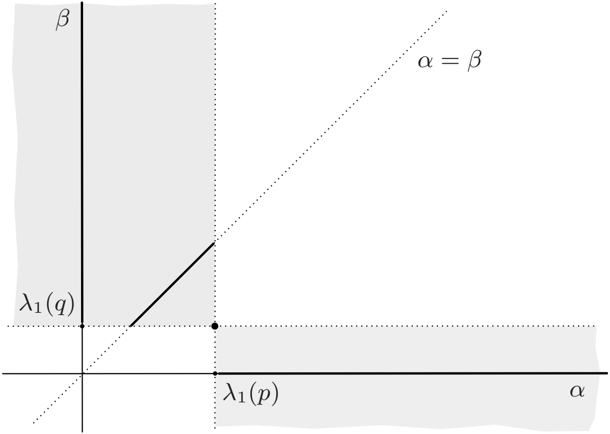
Two-parametric point of view

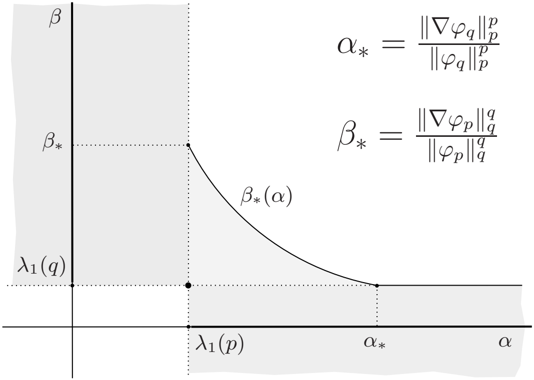


What happens for $\lambda \geqslant \max\{\lambda_1(q), \lambda_1(p)\}$, or, more general, for $\alpha \geqslant \lambda_1(p)$ and $\beta \geqslant \lambda_1(q)$?

First critical curve

In order to handle the existence of ground states of $E_{\alpha,\beta}$ in the case $\alpha \geqslant \lambda_1(p)$ and $\beta \geqslant \lambda_1(q)$, we define the following family of critical points:

$$\beta_*(\alpha) := \inf \left\{ \frac{\|\nabla u\|_q^q}{\|u\|_q^q} \, : \, u \in W_0^{1,p} \setminus \{0\} \text{ and } \frac{\|\nabla u\|_p^p}{\|u\|_p^p} \leqslant \alpha \right\}.$$



$$\alpha_* = \frac{\|\nabla \varphi_q\|_p^p}{\|\varphi_q\|_p^p}$$

$$\beta_* = \frac{\|\nabla \varphi_p\|_q^q}{\|\varphi_p\|_q^q}$$

Existence

Theorem

Let $\alpha > \lambda_1(p)$ and $\lambda_1(q) < \beta \leqslant \beta_*(\alpha)$. Then $(\mathcal{D}_{\alpha,\beta})$ has at least two positive solutions u_1 and u_2 such that

- $E_{\alpha,\beta}(u_1) < 0$ *and u_1 is the least energy solution (ground state), i.e.,*

$$E_{\alpha,\beta}(u_1) \leqslant E_{\alpha,\beta}(w) \quad \text{for any other solution } w \text{ of } (\mathcal{D}_{\alpha,\beta}).$$

- $E_{\alpha,\beta}(u_2) > 0$ *if* $\beta < \beta_*(\alpha)$, *and* $E_{\alpha,\beta}(u_2) = 0$ *if* $\beta = \beta_*(\alpha)$. *Moreover, u_2 is the least positive energy solution, i.e.,*

$$0 \leqslant E_{\alpha,\beta}(u_2) \leqslant E_{\alpha,\beta}(w) \quad \text{for any other solution } w \text{ of } (\mathcal{D}_{\alpha,\beta}) \text{ such that } E_{\alpha,\beta}(w) > 0.$$

However, recent results of [Il'yasov, Silva, 2017] indicate that $\beta_*(\alpha)$ is not a threshold curve for the existence of positive solutions of $(\mathcal{D}_{\alpha,\beta})$.

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Beyond $\beta_*(\alpha)$. Second critical curve

Define the following family of critical points:

$$\beta_{ps}(\alpha) := \sup\{\beta \in \mathbb{R} \,:\, (\mathcal{D}_{\alpha,\beta}) \text{ has a positive solution}\}$$

for $\alpha \geqslant \lambda_1(p)$.

Proposition

$\beta_*(\alpha) \leqslant \beta_{ps}(\alpha) < +\infty$ for any $\alpha \geqslant \lambda_1(p)$.

Main ingredient of the proof is the following generalized Picone's identity:

Lemma

Let $1 < q < p < \infty$. Then there exists $\rho > 0$ such that for any differentiable functions $u > 0$ and $\varphi \geqslant 0$ in Ω it holds

$$\left(|\nabla u|^{p-2} + |\nabla u|^{q-2}\right) \nabla u \nabla \left(\frac{\varphi^p}{u^{p-1} + u^{q-1}}\right) \leqslant \frac{|\nabla \varphi|^p + |\nabla \left(\varphi^{p/q}\right)|^q}{\rho}.$$

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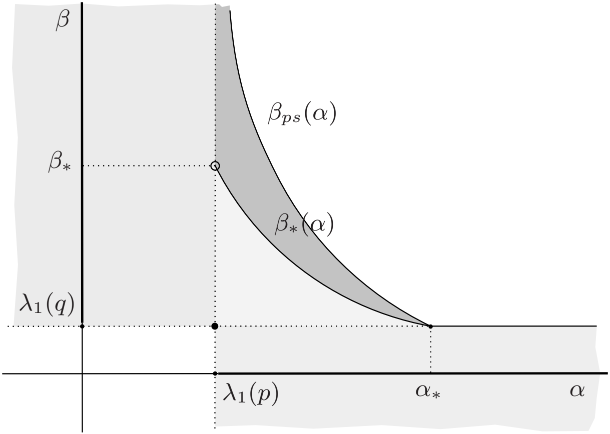
Existence. Properties of $\beta_{ps}(\alpha)$

Theorem

Let $\alpha \geqslant \lambda_1(p)$ and $\lambda_1(q) < \beta < \beta_{ps}(\alpha)$. Then $(\mathcal{D}_{\alpha,\beta})$ has a (nontrivial) positive solution.



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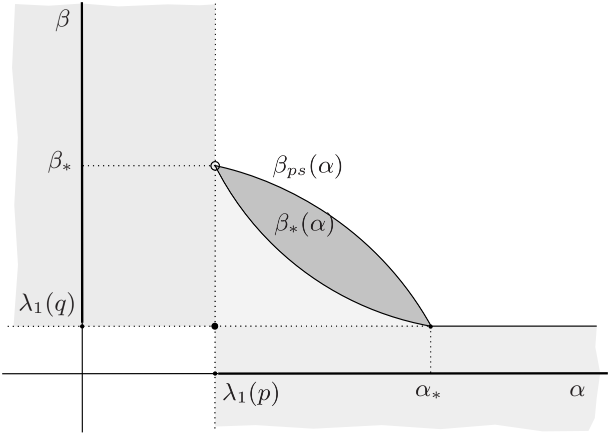

$\beta \neq \beta_*$

Recall that

$$\lambda_1(p) = \frac{\|\nabla \varphi_p\|_p^p}{\|\varphi_p\|_p^p}, \qquad \beta_* = \frac{\|\nabla \varphi_p\|_q^q}{\|\varphi_p\|_q^q}.$$

Lemma

Let $\alpha = \lambda_1(p)$.

- *If* $\lambda_1(q) < \beta < \beta_*$, *then* $E_{\alpha,\beta}$ *has a global minimizer.*
- *If* $\beta > \beta_*$, *then* $\inf\limits_{u \in W_0^{1,p}} E_{\alpha,\beta} = -\infty$.

$\beta = \beta_*$. A Fredholm-type result

Recall that

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Theorem

Let $\alpha = \lambda_1(p)$ and $\beta = \beta_*$.

- *If $p < 2q$, then $\inf_{u \in W_0^{1,p}} E_{\alpha,\beta} = -\infty$.*
- *If $p = 2q$, then $\inf_{u \in W_0^{1,p}} E_{\alpha,\beta} > -\infty$.*
- *If $p > 2q$, then $E_{\alpha,\beta}$ has a global minimizer.*

The situation is reminiscent of the Fredholm alternative for the p-Laplacian at the first eigenvalue, where the geometry of the energy functional (and hence the existence of its critical points) is crucially different for $p < 2$, $p = 2$, and $p > 2$; see, e.g., [Drábek, 2002], [Takáč, 2002] and references therein.

Fibered functional

Let us denote, for simplicity,

$$H_\alpha(u) := \int_\Omega |\nabla u|^p \, dx - \alpha \int_\Omega |u|^p \, dx,$$
$$G_\beta(u) := \int_\Omega |\nabla u|^q \, dx - \beta \int_\Omega |u|^q \, dx.$$

Consider

$$t(u) = \frac{|G_\beta(u)|^{\frac{1}{p-q}}}{|H_\alpha(u)|^{\frac{1}{p-q}}}.$$

Assume that $H_\alpha(u) \neq 0$. Then $t(u)u$ is a critical point of $E_{\alpha,\beta}$ if and only if u is a critical point of the fibered functional

$$J_{\alpha,\beta}(u) := -\mathrm{sign}(H_\alpha(u)) \, \frac{p-q}{pq} \, \frac{|G_\beta(u)|^{\frac{p}{p-q}}}{|H_\alpha(u)|^{\frac{q}{p-q}}}.$$

Note that $E_{\alpha,\beta}(t(u)u) = J_{\alpha,\beta}(u)$.

$p < 2q$

Take any $\theta \in C_0^\infty(\Omega)$ such that $\left\langle G'_\beta(\varphi_p), \theta \right\rangle < 0$. Consider the function $\varphi_p + \varepsilon\theta$.

According to the mean value theorem, there exist $\varepsilon_1 \in (0, \varepsilon)$ and $\varepsilon_2 \in (0, \varepsilon)$ such that

$$0 < H_\alpha(\varphi_p + \varepsilon\theta) = \varepsilon \langle H'_\alpha(\varphi_p + \varepsilon_1\theta), \theta \rangle \leqslant C\varepsilon^2 \quad \text{for } p \geqslant 2,$$
$$G_\beta(\varphi_p + \varepsilon\theta) = \varepsilon \langle G'_\beta(\varphi_p + \varepsilon_2\theta), \theta \rangle \leqslant -C\varepsilon < 0.$$

Using the fibered functional, we obtain

$$\inf_{u \in W_0^{1,p}} J_{\alpha,\beta}(u) \leqslant -\frac{p-q}{pq} \frac{|G_\beta(\varphi_p + \varepsilon\theta)|^{\frac{p}{p-q}}}{|H_\alpha(\varphi_p + \varepsilon\theta)|^{\frac{q}{p-q}}} \leqslant -C\, \varepsilon^{\frac{p}{p-q} - \frac{2q}{p-q}} \longrightarrow -\infty$$

as $\varepsilon \to 0$, whenever $p < 2q$. □

$p \geqslant 2q$

Suppose that $\inf_{u \in W_0^{1,p}} J_{\alpha,\beta}(u) = -\infty$. Let $\{u_n\}$ be the corresponding minimizing sequence. Let us make the L^2-decomposition $u_n = \tau_n \varphi_p + \tilde{u}_n$.

Using the improved Poincaré inequality from [Fleckinger-Pellé, Takáč, 2002], we have

$$H_\alpha(u_n) \geqslant C \left(\int_\Omega |\nabla \varphi_p|^{p-2} |\nabla \tilde{u}_n|^2 \, dx + \int_\Omega |\nabla \tilde{u}_n|^p \, dx \right).$$

On the other hand,

$$|G_\beta(u_n)| \leqslant C \left(\int_\Omega |\nabla \varphi_p|^{p-2} |\nabla \tilde{u}_n|^2 \, dx + \int_\Omega |\nabla \tilde{u}_n|^p \, dx \right)^{\frac{1}{2}}.$$

Therefore, noting that $\tilde{u}_n \to 0$ in $W_0^{1,p}$, we get

$$\liminf_{n\to\infty} J_{\alpha,\beta}(u_n) \geqslant -C \limsup_{n\to\infty} \left(\int_\Omega |\nabla \varphi_p|^{p-2} |\nabla \tilde{u}_n|^2 \, dx + \int_\Omega |\nabla \tilde{u}_n|^p \, dx \right)^{\frac{p-2q}{2(p-q)}} > -\infty$$

as $n \to \infty$ since $p \geqslant 2q$. A contradiction. □

References

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Thank you for your attention!