On some upper estimates for the first eigenvalue of a Sturm-Liouville problem in a weighted space

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Consider the Sturm-Liouville problem

$$y'' + Q(x)y + \lambda y = 0, \quad x \in (0,1), \tag{1}$$

$$y(0) = y(1) = 0, \tag{2}$$

where Q belongs to the set $T_{\alpha,\beta,\gamma}$ of all real-valued locally integrable functions on $(0,1)$ with non-negative values such that the following integral condition holds

$$\int_0^1 x^\alpha (1-x)^\beta Q^\gamma(x)dx = 1, \quad \alpha,\beta,\gamma \in \mathbb{R},\ \gamma \neq 0. \tag{3}$$

We give estimates for

$$M_{\alpha,\beta,\gamma} = \sup_{Q \in T_{\alpha,\beta,\gamma}} \lambda_1(Q).$$

For any function $Q \in T_{\alpha,\beta,\gamma}$ by H_Q we denote the closure of the space $C_0^\infty(0,1)$ in the norm $\|y\|_{H_Q} = \left(\int_0^1 y'^2 dx + \int_0^1 Q(x)y^2 dx\right)^{\frac{1}{2}}$.

For any function $Q \in T_{\alpha,\beta,\gamma}$ we can prove (see, for example, [1]) that

$$\lambda_1(Q) = \inf_{y \in H_Q \setminus \{0\}} R[Q,y], \quad \text{where } R[Q,y] = \frac{\int_0^1 \left(y'^2 - Q(x)y^2\right) dx}{\int_0^1 y^2 dx}.$$

Theorem 1. 1. *For any α,β,γ, $\gamma \neq 0$, we have $M_{\alpha,\beta,\gamma} \leqslant \pi^2$.*

2. *If $\gamma < 0$ or $0 < \gamma < 1$ and $\alpha,\beta > 3\gamma - 1$ then $M_{\alpha,\beta,\gamma} < \pi^2$. If $\gamma < -1$, $\alpha,\beta > 2\gamma-1$, $\alpha+\beta > 3\gamma-2$, then there exist a function $Q_* \in T_{\alpha,\beta,\gamma}$ and a positive on the interval $(0,1)$ function $u \in H_{Q_*}$ such that $M_{\alpha,\beta,\gamma} = R[Q_*,u]$, moreover u satisfies the equation*

$$u'' + mu = -x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}\, u^{\frac{\gamma+1}{\gamma-1}}$$

and the integral condition $\int_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}\, u^{\frac{2\gamma}{\gamma-1}}\, dx = 1$.

3. *If $\gamma > 0$ and $-1 < \alpha \leqslant 2\gamma - 1$, $-\infty < \beta < +\infty$ or $-1 < \beta \leqslant 2\gamma - 1$, $-\infty < \alpha < +\infty$ then $M_{\alpha,\beta,\gamma} = \pi^2$.*

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References

[1] S. Ezhak, E. Karulina, M. Telnova, *Estimates for the first eigenvalue of some Sturm-Liouville problems with an integral condition on the potential.* In: Astashova I. V. (ed.) Qualitative Properties of Solutions to Differential Equations and Related Topics of Spectral Analysis: scientific edition, M.: UNITY-DANA, (2012), 506–647. (Russian)