Eigenvalues and eigenfunctions of the elliptic boundary value problem with parameter

Alexey Filinovskiy

Moscow, Russian Federation

Consider the real separable Hilbert space H with the scalar product $(\cdot,\cdot)$ and the norm $\|\cdot\|$. Let e and g be non-zero elements of H. The number $\delta(e,g)=\sqrt{1-\frac{(e,g)^2}{\|e\|^2\|g\|^2}}$ is called the *deviation* between e and g. Let A be a linear compact positive operator in H. Denote by $\{\mu_j^A\}_{j=1}^{\infty}$ the sequence of its eigenvalues enumerated in the decreasing order and by $\{e_j^A\}_{j=1}^{\infty}$ the orthogonal basis of corresponding normalized eigenvectors. The number $\varrho_k^A=\inf\limits_{j\neq k}|\mu_j^A-\mu_k^A|$ is called the *isolation measure* of the eigenvalue μ_k^A.

Lemma 1. *Let $z\in H$ and $z\neq 0$. Then for all $k=1,2,\dots$ we have the inequalities $\|(A-\mu_k^A I)z\|\geq \varrho_k^A\delta(z,e_k^A)\|z\|$.*

Theorem 2. *Let A and B are linear compact positive operators and for some k we have $\max\{\varrho_k^A,\varrho_k^B\}>0$. Then the estimate $\delta(e_k^A,e_k^B)\leq \frac{2}{\max\{\varrho_k^A,\varrho_k^B\}}\|A-B\|$ holds.*

Now, consider the Robin eigenvalue problem $\sum_{i,j=1}^n(a_{ij}(x)u_{x_i})_{x_j}+\lambda u=0$, $x\in\Omega$, $\frac{\partial u}{\partial N}+\alpha u=0$, $x\in\partial\Omega$, in the bounded domain $\Omega\subset\mathbb{R}^n$, $n\geq 2$, with the boundary $\partial\Omega$ of C^3 class. The real coefficients $a_{ij}(x)\in C^2(\overline{\Omega})$ satisfy the symmetry condition $a_{ij}=a_{ji}$ and the ellipticity condition $\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j\geq\theta\sum_{i=1}^n\xi_i^2$, $(\xi_1,\dots,\xi_n)\in\mathbb{R}^n$, $x\in\Omega$, $\theta>0$. Here $\frac{\partial u}{\partial N}=\sum_{i,j=1}^n a_{ij}u_{x_i}\nu_j$, where $(\nu_1,\dots,\nu_n)$ is the outward unit normal vector to $\partial\Omega$, α is a real parameter. Let $\{\lambda_k^R(\alpha)\}_{k=1}^{\infty}$ be the sequence of its eigenvalues and $\{\lambda_k^D\}_{k=1}^{\infty}$ be the sequence of eigenvalues of the Dirichlet problem $\sum_{i,j=1}^n(a_{ij}(x)u_{x_i})_{x_j}+\lambda u=0$, $x\in\Omega$, $u=0$, $x\in\partial\Omega$ (enumerated in the increasing order according to their multiplicities). Denote by $\{u_{k,\alpha}^R(x)\}_{k=1}^{\infty}$ and $\{u_k^D(x)\}_{k=1}^{\infty}$ orthogonal normalized in $L_2(\Omega)$ sets of corresponding eigenfunctions. For all $\alpha\in\mathbb{R}$ we suppose that $\int_\Omega u_{k,\alpha}^R u_k^D\,dx\geq 0$. Denote by $m(\lambda)$ the multiplicity of the eigenvalue λ.

Theorem 3. *Let $m(\lambda_k^D)=1$. Then the eigenvalue $\lambda_k^R(\alpha)$ obeys an asymptotic expansion $\lambda_k^R(\alpha)=\lambda_k^D-\frac{\int_\Gamma\left(\frac{\partial u_k^D}{\partial N}\right)^2ds}{\int_\Omega(u_k^D)^2\,dx}\,\alpha^{-1}+o(\alpha^{-1})$, $\alpha\to+\infty$.*

Theorem 4. *Let $m(\lambda_k^D)=1$. Then there exists $\alpha_k\in\mathbb{R}$ such that for all $\alpha>\alpha_k$ we have $m(\lambda_k^R(\alpha))=1$ and the estimates $\frac{C_1}{\alpha}\leq\|u_{k,\alpha}^R-u_k^D\|_{H^2(\Omega)}\leq\frac{C_2}{\alpha}$ hold, where C_1, C_2 are positive constants independent of α.*

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References

[1] A. V. Filinovskiy, *On the estimates of eigenvalues of the boundary value problem with large parameter*, Tatra Mt. Math. Publ., **63** (2015), 101–113.

[2] A. V. Filinovskiy, *On the Asymptotic Behavior of Eigenvalues and Eigenfunctions of the Robin Problem with Large Parameter*, Math. Model. Anal., **22** (2017), 37–51.