Boundary value problems for families of functional differential equations

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We consider boundary value problems for linear functional solvability differential equations. Unimprovable sufficient solvability conditions will be obtained.

Let \( n \in \{1, 2, \ldots\} \), \( -\infty < b < a < +\infty \). We use the following standard notation: \( \mathbb{R} = (-\infty < b < a < +\infty) \), \( AC^{n-1}[a, b] \) is the space of functions \( x : [a, b] \to \mathbb{R} \) such that the functions \( x, \dot{x}, \ldots, x^{(n-1)} \) are absolutely continuous, \( C[a, b] \) is the space of continuous functions \( x : [a, b] \to \mathbb{R} \), \( L[a, b] \) is the space of integrable functions \( z : [a, b] \to \mathbb{R} \), an operator \( T \) from \( C[a, b] \) into \( L[a, b] \) is positive if \( Tx \) is almost everywhere non-negative for every non-negative \( x \in C[a, b] \), \( 1 \) is the unit function.

We find solutions of the following boundary value problems (1) and (3) in the space \( AC^{n-1}[a, b] \).

Suppose that \( q_i \in L[a, b], i = 0, \ldots, n-1 \), \( \ell_i : AC^{n-1}[a, b] \to \mathbb{R}, i = 1, \ldots, n \), are linear bounded functionals, \( f \in L[a, b], \alpha_i \in \mathbb{R}, i = 1, \ldots, n \).

Let non-negative functions \( p^+, p^- \in L[a, b] \) be given.

**Theorem 1.** If the boundary value problem

\[
\begin{align*}
  x^{(n)}(t) + \sum_{i=0}^{n-1} q_i(t)x^{(i)}(t) &= p_1(t)x(t_1) + p_2(t)x(t_2) + f(t), & t \in [a, b], \\
  \ell_ix &= \alpha_i, & i = 1, \ldots, n,
\end{align*}
\]  

has a unique solution for all functions \( p_1, p_2 \) and for all points \( t_1, t_2 \) such that

\[
\begin{align*}
  p_1, p_2 &\in L[a, b], p_1 + p_2 = p^+ - p^-, \quad -p^-(t) \leq p_i(t) \leq p^+(t), t \in [a, b], i = 1, 2, \\
  a \leq t_1 \leq t_2 \leq b,
\end{align*}
\]  

then the boundary value problem

\[
\begin{align*}
  x^{(n)}(t) + \sum_{i=0}^{n-1} q_i(t)x^{(i)}(t) &= (T^+x)(t) - (T^-x)(t) + f(t), & t \in [a, b], \\
  \ell_ix &= \alpha_i, & i = 1, \ldots, n,
\end{align*}
\]  

has a unique solution for all linear positive operators \( T^+, T^- : C[a, b] \to L[a, b] \) such that

\[
T^+1 = p^+, \quad T^-1 = p^-.
\]

Let \( M \) be a given subset of \( AC^{n-1}[a, b] \).

**Theorem 2.** Let the conditions of Theorem 1 be fulfilled. If the boundary value problem (1) has a solution from the set \( M \) for all functions \( p_1, p_2 \) and for all points \( t_1, t_2 \) such that conditions (2) are satisfied, then the boundary value problem (3) has a solution from the set \( M \) for all linear positive operators \( T^+, T^- : C[a, b] \to L[a, b] \) such that the equalities (4) are valid.

Various effective conditions for the solvability and the positive solvability of (3) can be obtained as corollaries of Theorems 1 and 2.

**Acknowledgement**

The work was performed as part of the State Task of the Ministry of Education and Science of the Russian Federation (project 1.5336.2017/BCH).

**2010 Mathematics Subject Classification:** 34K10.