Boundary value problems for families of functional differential equations

Eugene Bravyi

Perm′, Russia

We consider boundary value problems for linear functional differential equations. Unimprovable sufficient solvability conditions will be obtained.

Let $n \in \{1, 2, \ldots\}$, $-\infty < b < a < +\infty$. We use the following standard notation: $\mathbb{R} = (-\infty < b < a < +\infty)$, $\mathbf{AC}^{n-1}[a,b]$ is the space of functions $x : [a,b] \to \mathbb{R}$ such that the functions x, $\dot{x}$, …, $x^{(n-1)}$ are absolutely continuous, $\mathbf{C}[a,b]$ is the space of continuous functions $x : [a,b] \to \mathbb{R}$, $\mathbf{L}[a,b]$ is the space of integrable functions $z : [a,b] \to \mathbb{R}$, an operator T from $\mathbf{C}[a,b]$ into $\mathbf{L}[a,b]$ is positive if Tx is almost everywhere non-negative for every non-negative $x \in \mathbf{C}[a,b]$, $\mathbf{1}$ is the unit function. We find solutions of the following boundary value problems (1) and (3) in the space $\mathbf{AC}^{n-1}[a,b]$.

Suppose that $q_i \in \mathbf{L}[a,b]$, $i = 0, \ldots, n-1$, $\ell_i : \mathbf{AC}^{n-1}[a,b] \to \mathbb{R}$, $i = 1, \ldots, n$, are linear bounded functionals, $f \in \mathbf{L}[a,b]$, $\alpha_i \in \mathbb{R}$, $i = 1, \ldots, n$.

Let non-negative functions p^+, $p^- \in \mathbf{L}[a,b]$ be given.

Theorem 1. *If the boundary value problem*

$$\begin{cases} x^{(n)}(t) + \sum_{i=0}^{n-1} q_i(t)x^{(i)}(t) = p_1(t)x(t_1) + p_2(t)x(t_2) + f(t), \quad t \in [a,b], \\ \ell_i x = \alpha_i, \quad i = 1, \ldots, n, \end{cases} \tag{1}$$

has a unique solution for all functions p_1, p_2 and for all points t_1, t_2 such that

$$\begin{aligned} &p_1, p_2 \in \mathbf{L}[a,b],\ p_1 + p_2 = p^+ - p^-,\ -p^-(t) \le p_i(t) \le p^+(t), t \in [a,b],\ i = 1, 2, \\ &a \le t_1 \le t_2 \le b, \end{aligned} \tag{2}$$

then the boundary value problem

$$\begin{cases} x^{(n)}(t) + \sum_{i=0}^{n-1} q_i(t)x^{(i)}(t) = (T^+x)(t) - (T^-x)(t) + f(t), \quad t \in [a,b], \\ \ell_i x = \alpha_i, \quad i = 1, \ldots, n, \end{cases} \tag{3}$$

has a unique solution for all linear positive operators $T^+, T^- : \mathbf{C}[a,b] \to \mathbf{L}[a,b]$ such that

$$T^+\mathbf{1} = p^+, \quad T^-\mathbf{1} = p^-. \tag{4}$$

Let M be a given subset of $\mathbf{AC}^{n-1}[a,b]$.

Theorem 2. *Let the conditions of Theorem 1 be fulfilled. If the boundary value problem* (1) *has a solution from the set M for all functions p_1, p_2 and for all points t_1, t_2 such that conditions* (2) *are satisfied, then the boundary value problem* (3) *has a solution from the set M for all linear positive operators T^+, T^- : $\mathbf{C}[a,b] \to \mathbf{L}[a,b]$ such that the equalities* (4) *are valid.*

Various effective conditions for the solvability and the positive solvability of (3) can be obtained as corollaries of Theorems 1 and 2.

Acknowledgement

The work was performed as part of the State Task of the Ministry of Education and Science of the Russian Federation (project 1.5336.2017/BCH).

2010 Mathematics Subject Classification: 34K10.