Approximated analytical solutions of SW equations for particle-driven gravity currents propagating in channels of power-law cross-sections

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Differential Equations and Applications, September 4-7, 2017,

Brno,Czech

Introduction/Motivation

“Particle-driven gravity current” is the name of common effect: suspension of dense particles moves in horizontal direction into an ambient fluid because the densities are different.

Volcanic eruptions

Turbidity gravity currents:

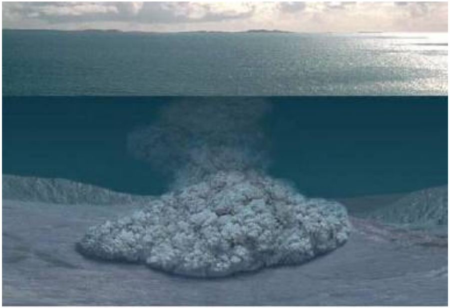
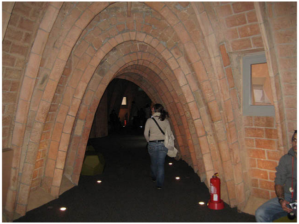


Image courtesy of the Open University

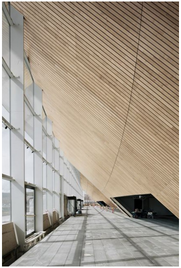
Introduction/Motivation: Examples

Parabolic corridor

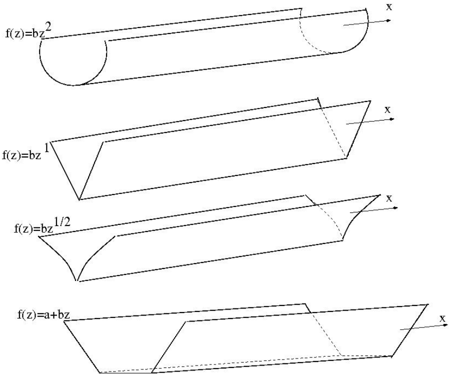


Valley

Triangle corridor



Schematic view - Typical channels



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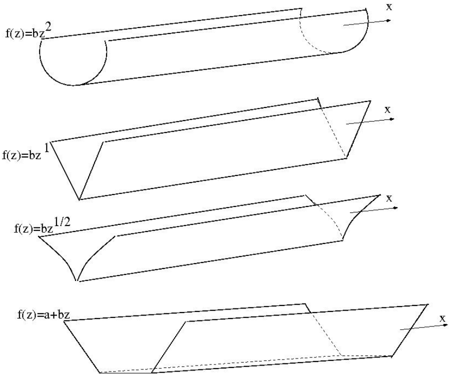

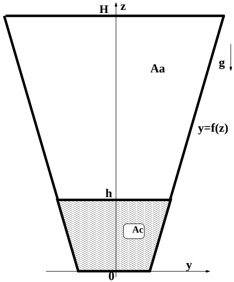

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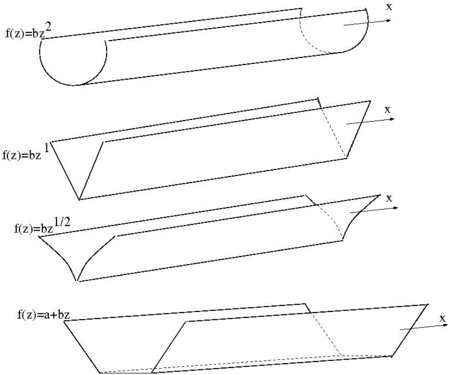
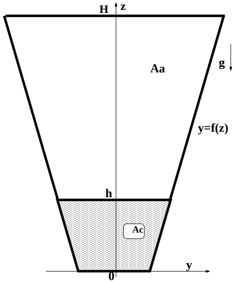
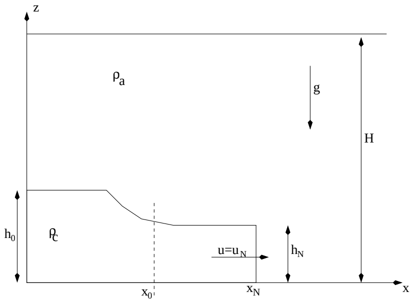




Cross-section area:

$$A_c(h) = \int_0^{h(x,t)} f(z)dz; \quad A_c'(h) = f(h); \quad A_T = \int_0^H f(z)dz$$

Schematic view - Side view



System parameters

Ambient fluid: density = ρ_a = const;

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Interstitial fluid: density = ρ_i; viscosity = ν

Particles:

- density ρ_p; radius a_p;
- Stokesian settling speed $W_s = \frac{2}{9}\frac{\rho_p - \rho_i}{\rho_i}\frac{a_p^2}{\nu}g$
- Volume fraction for concentration of the particles: $\kappa(x, z, t)$. Initially κ_0;

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Density of Current: $\rho_c = (1 - \kappa)\rho_i + \kappa \rho_p$;

Effective Reduced gravity : $g'_e = (\frac{\rho_c}{\rho_a} - 1)g = \varepsilon_p \kappa_0 \frac{\kappa}{\kappa_0} g$, where $\varepsilon_p = \frac{\rho_p - \rho_i}{\rho_i}$

Assumptions and Analysis

Assume: inviscid, $Re \gg 1$, Boussinesq, thin layer, negligible return flow

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Analysis: The pressures are hydrostatic and

$$\frac{\partial p_c}{\partial x} = \rho_a \varepsilon_p g \left[\kappa(x,t) \frac{\partial h}{\partial x} + (h - z) \frac{\partial \kappa}{\partial x} \right].$$

Governing equations and Front condition

* The volume continuity equation obtained using geometric considerations;

* The x-momentum equation is obtained by the usual averaging, plus the $\partial p_c/\partial x(x,t)$ term

* The "diffusion“ equation for volume fraction in the suspension

What happens at $x_N(t)$?

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What happens at $x_N(t)$?

$$u_N = (g'_e)^{1/2} h_N^{1/2} Fr(\varphi).$$

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$Fr(\varphi)$ is an “off the shelf” formula (Ungarish (2011)):

$$Fr = Fr_U(\varphi) = \left[\frac{2(1-\varphi)^2}{1+\varphi}(1+Q)\right]^{1/2},$$

where

$$\varphi = \frac{A_c(h)}{A_T}, \quad \text{and} \quad Q = \frac{\int_0^h z f(z) dz}{h \cdot [A_T - A_c(h)]}. \qquad (1)$$

SW equations

Scaling:

- $\{z,x,y\}$ w. $\{h_0,x_0,f(h_0)\}$ of lock;
- u w. $U=(\varepsilon_p\kappa_0 g h_0)^{1/2}$;
- t w. x_0/U; scaled volume fraction: $\phi=\frac{\kappa}{\kappa_0}$ in $[0,1]$
- scaled Stokes settling velocity of the particles: $\beta=\frac{W_s}{U}\frac{x_0}{h_0}$ $(\beta\ll 1)$

Equations of motion:

$$\begin{pmatrix} h_t \\ u_t \\ \phi_t \end{pmatrix} + \begin{pmatrix} u & \dfrac{h}{\alpha+1} & 0 \\ \phi & u & \frac{h}{\alpha+2} \\ 0 & 0 & u \end{pmatrix} \begin{pmatrix} h_x \\ u_x \\ \phi_x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\beta\phi\dfrac{\alpha+1}{h} \end{pmatrix}. \qquad (2)$$

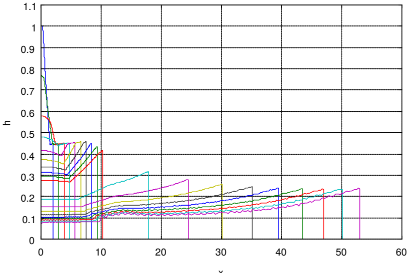
The system is hyperbolic; IC and BC are known

Finite-Difference results for $f(z) = z^{\alpha}$

Lax-Wendroff 2l method with 200 grid points: $\delta t = 10^{-3}$; $f(z) = z$;

$\beta = 2.5 * 10^{-3}$; $H = 10$

h vs. t : $t = 1(1)10(10), 100$



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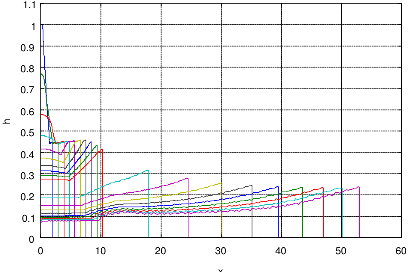

Slumping stage

* Constant u_N and h_N.

* x_s increases with α.

* As α increases:current faster;slumping stage -longer.

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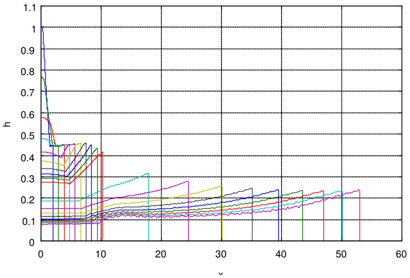

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Transient stage Decreasing height of the nose and speed.

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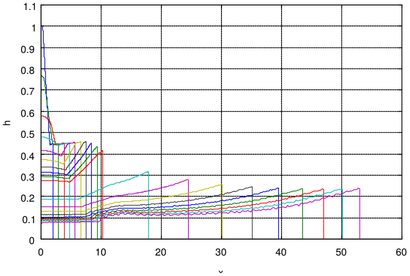

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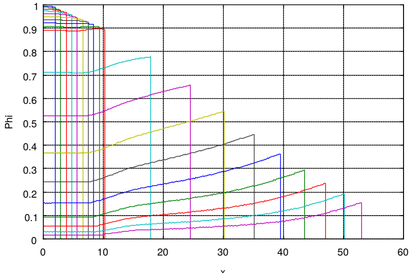
Self-Similarity Long-time profiles with "tail down - nose up" hight form.

Finite-Difference results for $f(z) = z^{\alpha}$

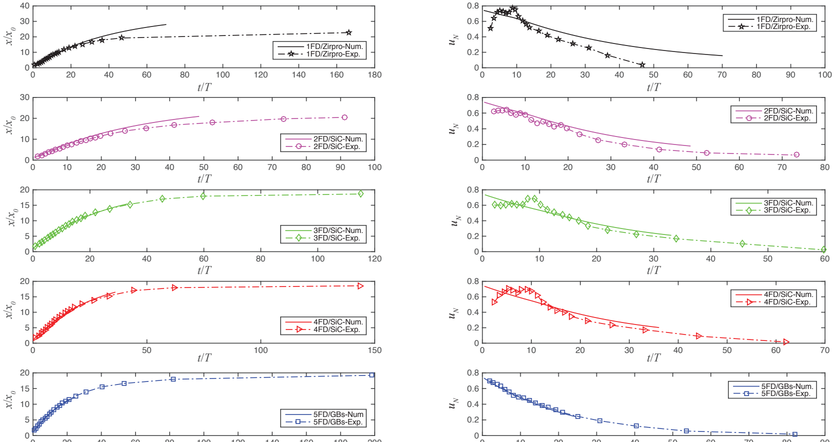
Volume fraction for $f(z) = z$

ϕ vs. t : $t = 1(1)10(10), 100$



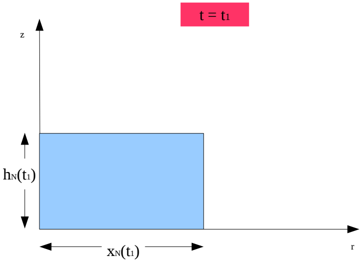
Comparison with Experiments

Experimental results of Mériaux et al.[2015] (symbols) and SW prediction (line) for $H = 1$, $f(z) = z$.



Box-Model

Box-model approximation: the current has a simple shape of length $x_N(t)$ and uniform height $h_N(t)$; $Fr = const$



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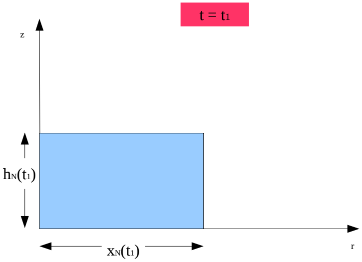

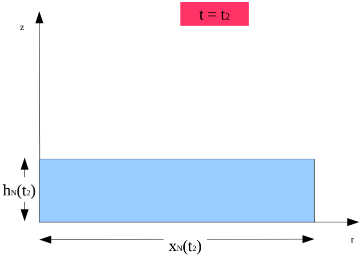

Box-Model Equations for power-law $f(z) = z^\alpha$

Conservation of the Particles and B.C.:

$$\begin{cases} \frac{d(V\phi)}{dt} = -\beta \phi x_N h_N^\alpha \\ \frac{dx_N}{dt} = Fr \phi^{1/2} h_N^{1/2} \end{cases} \quad (3)$$

Here

$$\begin{cases} V(t) = \int_0^{x_N} A(h) dx = h^\alpha x_N \\ Fr = const; \end{cases} \quad (4)$$

$V = V_0$ and ϕ decays.

$$x_{max}^{(T)} \approx (2\alpha + 5)^{\frac{2\alpha+2}{2\alpha+5}} (\alpha + 1)^{\frac{-2\alpha}{2\alpha+5}} \left(\frac{Fr}{\beta} \right)^{\frac{2\alpha+2}{2\alpha+5}} V_0^{\frac{3}{2\alpha+5}} \quad (5)$$

Similarity solution

- Large times after realease, the current "forgets" the initial conditions.

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$$x_N(t) = Kt^\gamma, \quad h_N(t,y) = (\dot{x}_N)^2 \mathscr{H}(y),$$
$$u_N(t,y) = \dot{x}_N \mathscr{U}(y),$$

where

$$y = \frac{x}{x_N};$$

And K and γ are constants.

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And K and γ are constants.

- The formulation above satisfies the nose condition $u_N = \dot{x}_N = Fr \cdot h_N^{1/2}$

Similarity solution for Homogeneous case

$$x_N = Kt^{\gamma}, \quad h = \gamma^2 K^2 t^{2\gamma-2} H(y), \quad u = \gamma K t^{\gamma-1} U(y), \quad \phi = 1,$$

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where

$$y = \frac{x}{x_N} = \frac{x}{Kt^{\gamma}}; \quad \gamma = \frac{2 + 2\alpha}{3 + 2\alpha}$$

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where

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$$U(y) = U_0(y) = y,$$
$$H(y) = H_0(y) = \frac{1}{Fr^2} - \frac{1}{4(\alpha+1)} + \frac{1}{4(\alpha+1)} \cdot y^2,$$

$$K = \left(\frac{1}{\gamma}\right)^{\gamma} \left(\frac{1}{\int_0^1 H^{\alpha+1}(y)dy} \right)^{1/(2\alpha+3)}$$

Similarity solution - Asymptotic analysis I

We use following expansions in the regime $\tau \ll 1$:

$$\begin{cases} x_N = Kt^{\gamma}(1+\tau X_1+\tau^2 X_2+...), \\ u = \gamma K t^{\gamma-1}(U_0(y)+\tau U_1(y)+\tau^2 U_2(y)+...), \\ h = \gamma^2 K^2 t^{2\gamma-2}(H_0(y)+\tau H_1(y)+\tau^2 H_2(y)+...), \\ \phi = 1+\tau\phi_1(y)+\tau^2\phi_2(y)+..., \end{cases}$$

where

$$\tau = \beta K^{-2} t^{3-2\gamma} = \beta K^{-2} t^{(5+2\alpha)/(3+2\alpha)}.$$

Similarity solution - Asymptotic analysis II

Substitution of expansions in equations of motion and b.c :

$$\left(\frac{1}{Fr^2} - \frac{1}{4(\alpha+1)} + \frac{y^2}{4(\alpha+1)}\right) H_1'' + \frac{1}{2} y H_1' - \frac{(2\alpha+5)(2\alpha+3)}{2(\alpha+1)} H_1 = F(y), \quad (6)$$

where $F(y)$ is function of y

$$F(y) = A\left(H_0'' + \alpha \frac{(H_0')^2}{H_0}\right). \quad (7)$$

The boundary conditions:

$$H_1'(0) = 0;$$

$$H_1(1) + \frac{(14+8\alpha-Fr^2)(\alpha+1)}{(5+2\alpha)(3+2\alpha)Fr^2} H_1'(1) = \frac{(3+2\alpha)^3}{4(5+2\alpha)(1+\alpha)} \left[1 + \frac{1}{2} \frac{(14+8\alpha-Fr^2)}{(3+2\alpha)(5+2\alpha}\right.$$

Similarity solution - Asymptotic analysis III

Upon the transformation of the independent variable,

$$\zeta = iy\left(\frac{4(\alpha+1)}{Fr^2} - 1\right)^{-1/2}, \tag{8}$$

(6) is reduced to

$$(1-\zeta^2)H_1'' - 2(\alpha+1)\zeta H_1' + 2(2\alpha+5)(2\alpha+3)H_1 = -4(\alpha+1)F(\zeta), \tag{9}$$

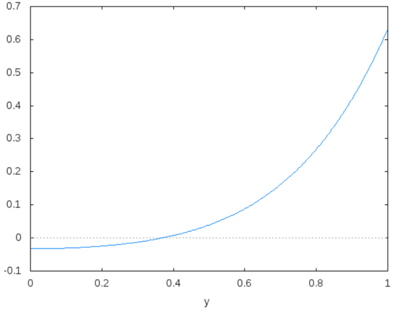
where $F(\zeta)$ is obtained from $F(y)$ by substitution of (8) into (7). This is a standard ultraspherical or Gegenbauer second-order differential equation. Its general solution is expressed by:

$$H_1(\zeta) = (\zeta^2-1)^{-\alpha/2}\left[C_1 P_{3\alpha+5}^{-\alpha}(\zeta) + C_2 Q_{3\alpha+5}^{-\alpha}(\zeta)\right] + H_1^p(\zeta), \tag{10}$$

where $P_{3\alpha+5}^{-\alpha}(\zeta)$ and $Q_{3\alpha+5}^{-\alpha}(\zeta)$ are associated Legendre functions of the first and second kinds. And $H_1^p(\zeta)$ is the particular solution of (9).

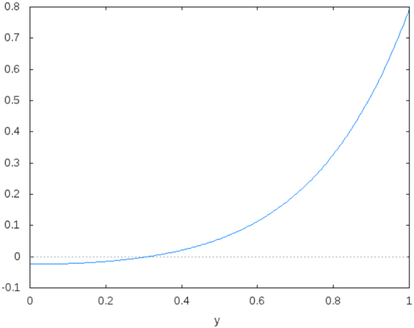
Asymptotcis similarity solutions $\alpha = 0$

$$H_1(\zeta) = -0.023 Q_5(\zeta) - \frac{9}{200} \tag{11}$$



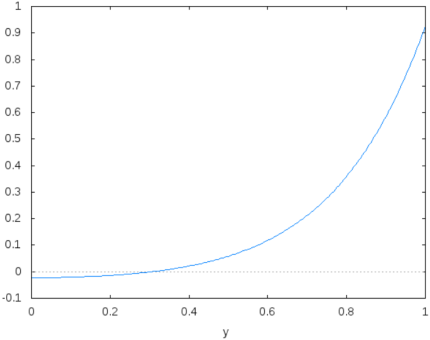
Asymptotcis similarity solutions $\alpha = 1$

$$H_1(\zeta) = -0.403(1-\zeta^2)^{-1/2} Q_8^{-1}(\zeta) + \frac{125}{1512} \cdot \frac{1}{1-\zeta^2} - \frac{25}{196} \qquad (12)$$



Asymptotcis similarity solutions $\alpha = 2$

$$H_1(\zeta) = -6.489 \cdot \frac{1}{1-\zeta^2} Q_{11}^{-2}(\zeta) - 0.041 + 0.17\zeta^2 \tag{13}$$

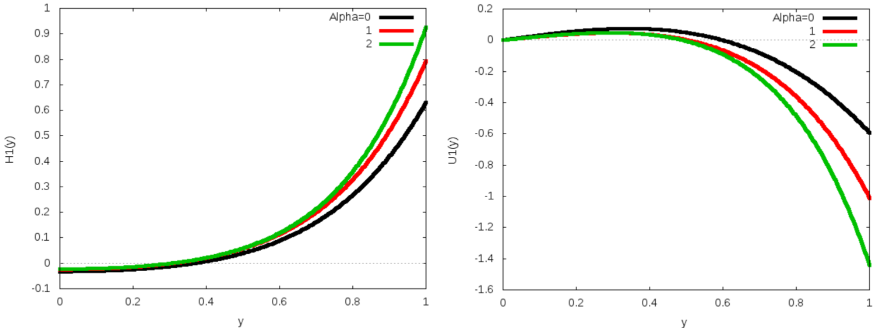


Asymptotcis similarity solutions $H_1(y)$ and $U_1(y)$

$$U_1(y) = -\frac{\beta}{2}\left[H_1'(y) - A\frac{H_0'}{H_0}\right], \tag{14}$$

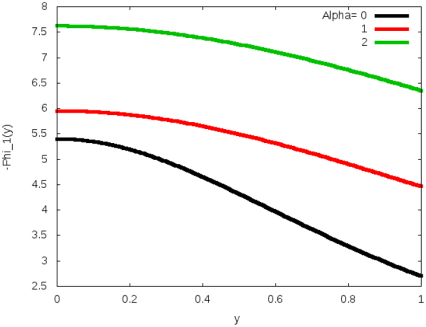
where

$$A = \frac{(\alpha+1)^2}{\alpha+2}\cdot\frac{1}{\beta^2(3-2\beta)} \tag{15}$$



Asymptotcis similarity solutions $\phi_1(y)$

$$\phi_1 = -\frac{(3+2\alpha)^3}{4(5+2\alpha)(1+\alpha)}/H_0(y). \tag{16}$$



Bi-Disperse currents

System parameters

Ambient fluid: density = ρ_a = const;

Interstitial fluid: density = ρ_i; viscosity = ν

Particles:

-2 types of particles

- Type 1: density $\rho_p^{(1)}$,radius $a_p^{(1)}$,concentration $\kappa^{(1)}(x,z,t)$ (init $\kappa_0^{(1)}$);
- Type 2: density $\rho_p^{(2)}$,radius $a_p^{(2)}$,concentration $\kappa^{(2)}(x,z,t)$ (init $\kappa_0^{(2)}$);
- Density ratio parameters: $\varepsilon_p^{(j)} = \dfrac{\rho_p^{(j)} - \rho_a}{\rho_a}$ $(j = 1,2)$.
- Stokesian settling speed $W_s^{(j)} = \frac{2}{9}\varepsilon_p^{(j)}\frac{(a_p^{(j)})^2}{\nu}g$, $(j = 1,2)$.

Density of Current: $\rho_c = \rho_a(1 + \kappa^{(1)}\varepsilon_p^{(1)} + \kappa^{(2)}\varepsilon_p^{(2)})$

Shallow-water equations of motion

$$
\begin{pmatrix} h \\ u \\ \phi^{(1)} \\ \phi^{(2)} \end{pmatrix}_t + \begin{pmatrix} u & \frac{h}{\alpha+1} & 0 & 0 \\ \phi^{(1)}+\phi^{(2)} & u & \frac{h}{\alpha+2} & \frac{h}{\alpha+2} \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & u \end{pmatrix} \begin{pmatrix} h \\ u \\ \phi^{(1)} \\ \phi^{(2)} \end{pmatrix}_x =
$$

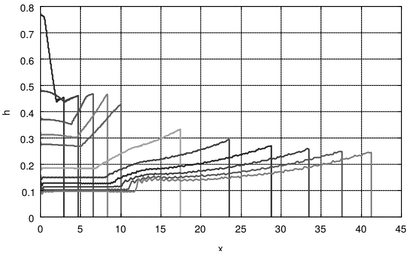
$$
= \begin{pmatrix} 0 \\ 0 \\ -\beta^{(1)}\phi^{(1)}\frac{\alpha+1}{h} \\ -\beta^{(2)}\phi^{(2)}\frac{\alpha+1}{h} \end{pmatrix}.
$$

Finite-Difference results for $f(z) = z^1$

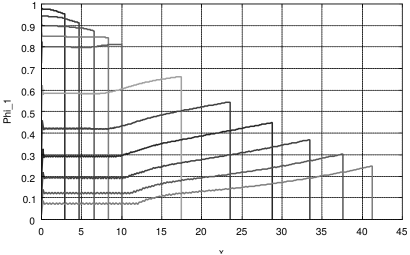
We use Lax-Wendroff 2l method with 200 grid points; $\delta t = 10^{-3}$; $f(z) = z$;

$H = 10$, $\beta_1 = 2.5 * 10^{-3}, \beta_2 = 7\beta_1, \phi_1(0) = 0.8$.

h vs. t : $t = 2(2)10, 20(10), 70$

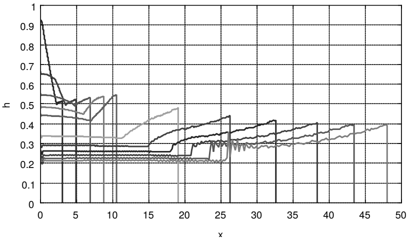


ϕ_1 vs. t : $t = 2(2)10, 20(10), 70$



Finite-Difference results for $f(z) = z^2$

h vs. t : $t = 2(2)10, 20(10), 70$



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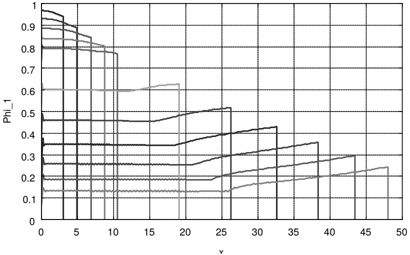
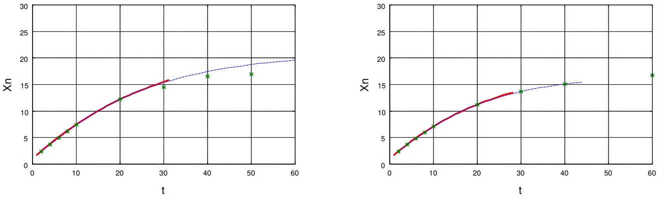


Comparison with Experiments

Experimental results of Mériaux et al.[2016] (symbols) and SW prediction (line) for triangle cross-section. $H = 1$, $f(z) = z$. Also shown the SW solution for average values of $\overline{\beta} = \Sigma_{j=1}^{2} \beta^{(j)} \phi^{(j)}(0)$. (dashed line).



Box-Model Runout for power-law $f(z) = z^\alpha$

$V = V_0$ and ϕ_1 decays.

$$x_{max} \approx \left(\frac{Fr(2\alpha+5)}{2\beta_1}\right)^{\frac{2\alpha+2}{2\alpha+5}} V_0^{\frac{3}{2\alpha+5}} (\alpha+1)^{-\frac{4\alpha+1}{2\alpha+5}} \cdot \int_0^{V_0\phi^{(1)}(0)} \frac{\left[\Sigma_{j=1}^2 a_j \cdot y^{\beta^{(j)}/\beta^{(1)}}\right]^{1/2}}{y} dy \tag{17}$$

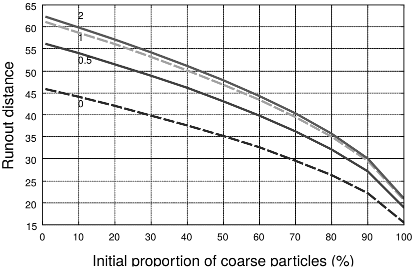
where

$$a_j = \frac{\phi^{(j)}(0)}{(\phi^{(1)}(0) \cdot V)^{\beta^{(j)}/\beta^{(1)}}}, ((j = 1, 2)) \tag{18}$$

Box-Model Prediction of Runout distance for $f(z) = z^{\alpha}$

$\alpha = 0, 0.5, 1, 2$; $H = 10, \beta_1 = 2.5 * 10^{-3}, \beta_2 = 7\beta_1$.

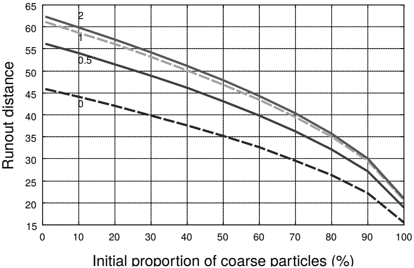
Runout decreases with increasing of coarse particles concentration



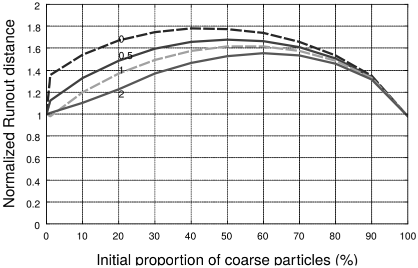
Box-Model Prediction of Runout distance for $f(z) = z^{\alpha}$

$\alpha = 0, 0.5, 1, 2$; $H = 10, \beta_1 = 2.5 * 10^{-3}, \beta_2 = 7\beta_1$.

Runout decreases with increasing of coarse particles concentration



Runout normalized by mono $\overline{\beta}$: Degree of polydispersion increases the runout .



Summary

- Mono-disperse currents: Analytical expression for runout distance and Analytical Similarity solution
- Poly-disperse currents: Analytical expression for runout distance


thank you