Energy decay and global nonexistence of solutions for a nonlinear hyperbolic equation

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Introduction

In this work we study the following quasilinear hyperbolic equations

$$\begin{cases} u_{tt} - \operatorname{div}\left(\left|\nabla u\right|^{m} \nabla u\right) - \triangle u_t + \left|u_t\right|^{q-1} u_t = \left|u\right|^{p-1} u, & x \in \Omega,\ t > 0, \\ u\left(x, 0\right) = u_0\left(x\right),\ u_t\left(x, 0\right) = u_1\left(x\right), & x \in \Omega, \\ u\left(x, t\right) = 0, & x \in \partial\Omega,\ t > 0 \end{cases}, \tag{1}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$ in R^n $(n \geq 1)$; $m > 0$, $p, q \geq 1$.

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When $m = 0$, (1) becomes the following wave equation with nonlinear and strong damping terms

$$u_{tt} - \triangle u - \triangle u_t + |u_t|^{q-1} u_t = |u|^{p-1} u. \qquad (2)$$

Gerbi and Houari studied the exponential decay, Chen and Liu studied global existence, decay and exponential growth of solutions of the problem (2). Also, Gazzola and Squassina studied global existence and blow up of solutions of the problem (2), for $q = 1$.

S. Gerbi, B.S. Houari, Exponential decay for solutions to semilinear damped wave equation, Discrete Continuous Dynamical Systems Series B, 5(3) 559-566 (2012).

H. Chen, G. Liu, Global existence, uniform decay and exponential growth for a class of semilinear wave equation with strong damping, Acta Mathematica Scienta, 33B(1) (2013).

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Introduction

In the absence of the strong damping term $\triangle u_t$, and $m = 0$, the problem (1) can be reduced to the following wave equation with nonlinear damping and source terms

$$u_{tt} - \triangle u + |u_t|^{q-1} u_t = |u|^{p-1} u. \tag{3}$$

Many authors have investigated the local existence, blow up and asymptotic behavior of solutions of the equation (3), see)Georgiev-Todorova 1994, Levine 1974, Messaoudi 2001, Runzhang-Jihong 2009, Vitillaro 1999).

V. Georgiev, G. Todorova, Existence of a solution of the wave equation with nonlinear damping and source terms, J. Differential Equations, 109(2) 295–308 (1994).

H.A. Levine, Instability and nonexistence of global solutions to nonlinear wave equations of the form $Pu_{tt} = Au + F(u)$, Trans. Amer. Math. Soc., 192, 1–21 (1974).

S.A. Messaoudi, Blow up in a nonlinearly damped wave equation,

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$$u_{tt} - \triangle u + |u_t|^{q-1} u_t = |u|^{p-1} u \qquad (3)$$

The interaction between damping $(|u_t|^{q-1} u_t)$ and the source term $(|u|^{p-1} u)$ makes the problem more interasting. Levine, first studied the interaction between the linear damping $(q = 1)$ and source term by using Concavity method. But this method can't be applied in the case of a nonlinear damping term. Georgiev and Todorova extended Levine's result to the nonlinear case $(q > 1)$. They showed that solutions with negative initial energy blow up in finite time. Later, Vitillaro extended these results to the case of nonlinear damping and positive initial energy.

Introduction

Messaoudi studied decay of solutions of the problem (1), using the techniques combination of the perturbed energy and potential well methods. Recenctly, the problem (1) was studied by Wu and Xue. They proved uniform energy decay rates of solutions, by utilizing the multiplier metdod.

S.A. Messaoudi, On the decay of solutions for a class of quasilinear hyperbolic equations with nonlinear damping and source terms, Math Meth Appl Sci, 28, 1819-1828 (2005).

Y. Wu, X. Xue, Uniform decay rate estimates for a class of quasilinear hyperbolic equations with nonlinear damping and source terms, Appl Anal, 92(6) 1169-1178 (2013).

Introduction

In this work, we established the polynomial and exponential decay of solutions of the problem (1) by using Nakao's inequality. After that, we show blow up of solutions with negative and nonnegative initial energy, using the same techniques as in (Li-Tsai 2003).

This work is organized as follows: In the next section, we present some lammas, notations and local existence theorem. In section 3, the global existence and decay of solutions are given. In the last section, we show the blow up of solutions, for $q = 1$.

Preliminaries

In this section, we shall give some assumptions and lemmas which will be used throughout this paper. Let $\|.\|$ and $\|.\|_p$ denote the usual $L^2(\Omega)$ norm and $L^p(\Omega)$ norm, respectively.

Lemma (1)

(Sobolev-Poincare inequality) (Adams, Fournier 2013). Let p be a number with $2 \leq p < \infty$ $(n = 1, 2)$ *or* $2 \leq p \leq \frac{2n}{n-2}$ $(n \geq 3)$ *, then there is a constant* $C_* = C_*(\Omega,\ p)$ *such that*

$$\|u\|_p \leq C_* \|\nabla u\| \text{ for } u \in H_0^1(\Omega).$$

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Lemma (2)

(Nakao 1977). Let $\phi(t)$ be nonincreasing and nonnegative function defined on $[0, T]$, $T > 1$, satisfying

$$\phi^{1+\alpha}(t) \leq w_0 \left(\phi(t) - \phi(t+1)\right), \ t \in [0, T]$$

for w_0 is a positive constant and α is a nonnegative constant. Then we have, for each $t \in [0, T]$,

$$\begin{cases} \phi(t) \leq \phi(0) e^{-w_1 [t-1]^+}, & \alpha = 0, \\ \phi(t) \leq \left(\phi(0)^{-\alpha} + w_0^{-1} \alpha [t-1]^+\right)^{-\frac{1}{\alpha}}, & \alpha > 0, \end{cases}$$

where $[t-1]^+ = \max\{t-1, 0\}$, *and* $w_1 = \ln\left(\frac{w_0}{w_0 - 1}\right)$.

Preliminaries

Next, we state the local existence theorem that can be established by combining arguments of (Georgiev-Todorova 1994, Pişkin 2013, Ye 2013).

Theorem (3)

(Local existence). Suppose that $m+2<p+1<\frac{n(m+2)}{n-(m+2)}$, $m+2<n$ *and further* $u_0 \in W_0^{1,m+2}(\Omega)$ *and* $u_1 \in L^2(\Omega)$ *such that problem (1) has a unique local solution*

$$u \in C\left([0,T);W_0^{1,m+2}(\Omega)\right) \text{ and } u_t \in C\left([0,T);L^2(\Omega)\right) \cap L^{q+1}(\Omega \times [0,T))$$

Moreover, at least one of the following statements holds true:
i) $T=\infty$,
ii) $\|u_t\|^2+\|\nabla u\|_{m+2}^{m+2} \longrightarrow \infty$ *as* $t \longrightarrow T^-$.

Global existence and decay of solutions

In this section, we discuss the global existence and decay of the solution for problem (1).
We define

$$J(t) = \frac{1}{m+2} \|\nabla u\|_{m+2}^{m+2} - \frac{1}{p+1} \|u\|_{p+1}^{p+1}, \tag{4}$$

and

$$I(t) = \|\nabla u\|_{m+2}^{m+2} - \|u\|_{p+1}^{p+1}. \tag{5}$$

We also define the energy function as follows

$$E(t) = \frac{1}{2} \|u_t\|^2 + \frac{1}{m+2} \|\nabla u\|_{m+2}^{m+2} - \frac{1}{p+1} \|u\|_{p+1}^{p+1}. \tag{6}$$

Finally, we define

$$W = \left\{u : u \in W_0^{1,m+2}(\Omega),\ I(u) > 0\right\} \cup \{0\}. \tag{7}$$

Global existence and decay of solutions

The next lemma shows that our energy functional (6) is a nonincreasing function along the solution of (1).

Lemma (4)

$E(t)$ is a nonincreasing function for $t \geq 0$ and

$$E'(t) = -\left(\|u_t\|_{q+1}^{q+1} + \|\nabla u_t\|^2 \right) \leq 0. \tag{8}$$

Proof. Multiplying the equation of (1) by u_t and integrating over Ω, using integrating by parts, we get

$$E(t) - E(0) = -\int_0^t \left(\|u_\tau\|_{q+1}^{q+1} + \|\nabla u_\tau\|^2 \right) d\tau \text{ for } t \geq 0. \tag{9}$$

Global existence and decay of solutions

Lemma (5)

Let $u_0 \in W$ and $u_1 \in L^2(\Omega)$. Suppose that $p > m+1$ and

$$\beta = C_* \left(\frac{(p+1)(m+2)}{p-m-1} E(0) \right)^{\frac{p-m-1}{m+2}} < 1, \qquad (10)$$

then $u \in W$ for each $t \geq 0$.

Proof. Since $I(0) > 0$, it follows the continuity of $u(t)$ that

$$I(t) > 0,$$

for some interval near $t = 0$. Let $T_m > 0$ be a maximal time, when (5) holds on $[0, T_m]$.

Global existence and decay of solutions

From (4) and (5), we have

$$\begin{aligned} J(t) &= \frac{1}{p+1} I(t) + \frac{p-m-1}{(p+1)(m+2)} \|\nabla u\|_{m+2}^{m+2} \\ &\geq \frac{p-m-1}{(p+1)(m+2)} \|\nabla u\|_{m+2}^{m+2}. \end{aligned} \tag{11}$$

Thus, from (6) and $E(t)$ being nonincreasing by (8), we have that

$$\begin{aligned} \|\nabla u\|_{m+2}^{m+2} &\leq \frac{(p+1)(m+2)}{p-m-1} J(t) \\ &\leq \frac{(p+1)(m+2)}{p-m-1} E(t) \\ &\leq \frac{(p+1)(m+2)}{p-m-1} E(0). \end{aligned} \tag{12}$$

Global existence and decay of solutions

And so, exploiting Lemma 1, (10) and (12), we obtain

$$\begin{aligned} \|u\|_{p+1}^{p+1} &\leq C_* \|\nabla u\|^{p+1} \\ &\leq C_* \|\nabla u\|_{m+2}^{p+1} \\ &= C_* \|\nabla u\|_{m+2}^{p-m-1} \|\nabla u\|_{m+2}^{m+2} \\ &\leq C_* \left(\frac{(p+1)(m+2)}{p-m-1} E(0) \right)^{\frac{p-m-1}{m+2}} \|\nabla u\|_{m+2}^{m+2} \\ &= \beta \|\nabla u\|_{m+2}^{m+2} \\ &< \|\nabla u\|_{m+2}^{m+2} \text{ on } t \in [0, T_m]. \qquad (13) \end{aligned}$$

Therefore, by using (5), we conclude that $I(t) > 0$ for all $t \in [0, T_m]$. By repeating the procedure, T_m is extended to T. The proof of Lemma 5 is completed.

Global existence and decay of solutions

Lemma (6)

Let assumptions of Lemma 5 holds. Then there exists $\eta_1 = 1 - \beta$ such that

$$\|u\|_{p+1}^{p+1} \leq (1 - \eta_1) \|\nabla u\|_{m+2}^{m+2}.$$

Proof. From (13), we get

$$\|u\|_{p+1}^{p+1} \leq \beta \|\nabla u\|_{m+2}^{m+2}.$$

Let $\eta_1 = 1 - \beta$, then we have the result.
Remark (7). From Lemma 6, we can deduce that

$$\|\nabla u\|_{m+2}^{m+2} \leq \frac{1}{\eta_1} I(t). \tag{14}$$

Global existence and decay of solutions

Theorem (8)

Suppose that $m+2 < p+1 < \frac{n(m+2)}{n-(m+2)}$, $m+2 < n$ *holds. Let* $u_0 \in W$ *satisfying (10). Then the solution of problem (1) is global.*

Proof. It is sufficient to show that $\|u_t\|^2 + \|\nabla u\|_{m+2}^{m+2}$ is bounded independently of t. To achieve this we use (5) and (6) to obtain

$$\begin{aligned} E(0) &\geq E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{m+2}\|\nabla u\|_{m+2}^{m+2} - \frac{1}{p+1}\|u\|_{p+1}^{p+1} \\ &= \frac{1}{2}\|u_t\|^2 + \frac{p-m-1}{(p+1)(m+2)}\|\nabla u\|_{m+2}^{m+2} + \frac{1}{p+1}I(t) \\ &\geq \frac{1}{2}\|u_t\|^2 + \frac{p-m-1}{(p+1)(m+2)}\|\nabla u\|_{m+2}^{m+2} \end{aligned}$$

since $I(t) \geq 0$.

Global existence and decay of solutions

Therefore

$$\left\| u_t \right\|^2 + \left\| \nabla u \right\|_{m+2}^{m+2} \leq CE\left(0\right),$$

where $C = \max\left\{2, \frac{(p+1)(m+2)}{p-m-1}\right\}$. Then by Theorem 3, we have the global existence result.

Global existence and decay of solutions

Theorem (9)

Suppose that $m+2 < p+1 < \frac{n(m+2)}{n-(m+2)}$, $m+2 < n$ *and (10) hold, and further* $u_0 \in W$. *Thus, we have following decay estimates:*

$$E(t) \leq \begin{cases} E(0)\, e^{-w_1 [t-1]^+}, & \text{if } q = 1, \ m = 0 \\ \left(E(0)^{-\alpha} + C_7^{-1} \alpha \, [t-1]^+ \right)^{-\frac{1}{\alpha}}, & \text{if } q > \frac{1}{m+1}, \end{cases}$$

where w_1, α *and* C_7 *are positive constants which will be defined later.*

Global existence and decay of solutions

Proof. By integrating (8) over $[t, t+1]$, $t > 0$, we have

$$E(t) - E(t+1) = D^{q+1}(t), \tag{15}$$

where

$$D^{q+1}(t) = \int_t^{t+1} \left(\|u_\tau\|_{q+1}^{q+1} + \|\nabla u_\tau\|^2 \right) d\tau. \tag{16}$$

By virtue of (16) and Hölder's inequality, we observe that

$$\int_t^{t+1} \int_\Omega |u_t|^2 \, dxdt \leq |\Omega|^{\frac{q+1}{q+2}} D^2(t) = CD^2(t). \tag{17}$$

Global existence and decay of solutions

Hence, from (17), there exist $t_1 \in \left[t, t+\frac{1}{4}\right]$ and $t_2 \in \left[t+\frac{3}{4}, t+1\right]$ such that

$$\|u_t(t_i)\| \leq CD(t), \quad i = 1, 2 \tag{18}$$

Multiplying (1) by u, and integrating it over $\Omega \times [t_1, t_2]$, using integration by parts, we get

$$\begin{aligned} \int_{t_1}^{t_2} I(t)\, dt &= -\int_{t_1}^{t_2} \int_\Omega u u_{tt}\, dxdt - \int_{t_1}^{t_2} \int_\Omega \nabla u_t \nabla u\, dxdt \\ &\quad - \int_{t_1}^{t_2} \int_\Omega |u_t|^{q-1} u_t u\, dxdt. \end{aligned} \tag{19}$$

Global existence and decay of solutions

By using (1) and integrating by parts and Cauchy-Schwarz inequality in the first term, and Hölder inequality in the second term of the right hand side of (19), we obtain

$$\begin{aligned} \int_{t_1}^{t_2} I(t)\,dt \quad \leq \quad & \|u_t(t_1)\| \|u(t_1)\| + \|u_t(t_2)\| \|u(t_2)\| \\ & + \int_{t_1}^{t_2} \|u_t(t)\|^2 dt + \int_{t_1}^{t_2} \|\nabla u_t\| \|\nabla u\| dt \\ & - \int_{t_1}^{t_2} \int_\Omega |u_t|^{q-1} u_t u dx dt. \end{aligned} \tag{20}$$

Now, our goal is to estimate the last term in the right-hand side of inequality (20). By using Hölder inequality, we obtain

$$\int_{t_1}^{t_2} \int_\Omega |u_t|^{q-1} u_t u dx dt \leq \int_{t_1}^{t_2} \|u_t(t)\|_{q+1}^{q} \|u(t)\|_{q+1} dt \tag{21}$$

Global existence and decay of solutions

By applying the Sobolev-Poincare inequality and (12), we find

$$
\begin{aligned}
& \int_{t_1}^{t_2} \|u_t(t)\|_{q+1}^{q} \|u(t)\|_{q+1} dt \\
\leq\ & C_* \int_{t_1}^{t_2} \|u_t(t)\|_{q+1}^{q} \|\nabla u\| dt \\
\leq\ & C_* \int_{t_1}^{t_2} \|u_t(t)\|_{q+1}^{q} \|\nabla u\|_{m+2} dt \\
\leq\ & C_* \left(\frac{(p+1)(m+2)}{p-m-1} E(0) \right)^{\frac{1}{m+2}} \int_{t_1}^{t_2} \|u_t(t)\|_{q+1}^{q} E^{\frac{1}{m+2}}(s) dt \\
\leq\ & C_* \left(\frac{(p+1)(m+2)}{p-m-1} E(0) \right)^{\frac{1}{m+2}} \sup_{t_1 \leq s \leq t_2} E^{\frac{1}{m+2}}(s) \int_{t_1}^{t_2} \|u_t(t)\|_{q+1}^{q} dt \\
\leq\ & C_* \left(\frac{(p+1)(m+2)}{p-m-1} E(0) \right)^{\frac{1}{m+2}} \sup_{t_1 \leq s \leq t_2} E^{\frac{1}{m+2}}(s) D^q(t). \qquad (22)
\end{aligned}
$$

Global existence and decay of solutions

Now, we estimate the fourth term of the right-hand side of inequality (20). By using the embedding $L^{m+2}(\Omega) \hookrightarrow L^2(\Omega)$, we have

$$\begin{aligned}
&\int_{t_1}^{t_2} \|\nabla u_t\| \|\nabla u\| dt \\
\leq\ & C_* \int_{t_1}^{t_2} \|\nabla u_t\| \|\nabla u(t)\|_{m+2} dt \\
\leq\ & C_* \left(\frac{(p+1)(m+2)}{p-m-1} E(0) \right)^{\frac{1}{m+2}} \int_{t_1}^{t_2} \|\nabla u_t\| E^{\frac{1}{m+2}}(s) dt \\
\leq\ & C_* \left(\frac{(p+1)(m+2)}{p-m-1} E(0) \right)^{\frac{1}{m+2}} \sup_{t_1 \leq s \leq t_2} E^{\frac{1}{m+2}}(s) \int_{t_1}^{t_2} \|\nabla u_t\| dt,
\end{aligned}$$

which implies

$$\begin{aligned}
\int_{t_1}^{t_2} \|\nabla u_t\| dt &\leq \left(\int_{t_1}^{t_2} 1 dt \right)^{\frac{1}{2}} \left(\int_{t_1}^{t_2} \|\nabla u_t\|^2 dt \right)^{\frac{1}{2}} \\
&\leq CD(t)
\end{aligned}$$

Global existence and decay of solutions

Then

$$\int_{t_1}^{t_2} \|\nabla u_t\| \, \|\nabla u\| \, dt \leq CC_* \left(\frac{(p+1)(m+2)}{p-m-1} E(0) \right)^{\frac{1}{m+2}} \sup_{t_1 \leq s \leq t_2} E^{\frac{1}{m+2}}(s) \, D(t) \tag{23}$$

From (12), (18) and Sobolev-Poincare inequality, we have

$$\|u_t(t_i)\| \, \|u(t_i)\| \leq C_1 D(t) \sup_{t_1 \leq s \leq t_2} E^{\frac{1}{m+2}}(s), \tag{24}$$

where $C_1 = 2C_* \left(\frac{(p+1)(m+2)}{p-m-1} E(0) \right)^{\frac{1}{m+2}}$. Then by (20)-(24) we have

$$\begin{aligned} \int_{t_1}^{t_2} I(t) \, dt \quad \leq \quad & C_1 D(t) \sup_{t_1 \leq s \leq t_2} E^{\frac{1}{m+2}}(s) + D^2(t) \\ & + CC_* \left(\frac{(p+1)(m+2)}{p-m-1} E(0) \right)^{\frac{1}{m+2}} \sup_{t_1 \leq s \leq t_2} E^{\frac{1}{m+2}}(s) \, D(t) \\ & + C \left(\frac{(p+1)(m+2)}{p-m-1} E(0) \right)^{\frac{1}{m+2}} \sup E^{\frac{1}{m+2}}(s) \, D^q \quad (25) \end{aligned}$$

Global existence and decay of solutions

On the other hand, from (5), (6) and Remark 7, we obtain

$$E(t) \leq \frac{1}{2}\|u_t\|^2 + C_3 I(t), \tag{26}$$

where $C_3 = \frac{1}{\eta_1}\frac{p-m-1}{(p+1)(m+2)} + \frac{1}{p+1}$. By integrating (26) over $[t_1, t_2]$, we have

$$\int_{t_1}^{t_2} E(t)\,dt \leq \frac{1}{2}\int_{t_1}^{t_2}\|u_t\|^2\,dt + C_3\int_{t_1}^{t_2} I(t)\,dt. \tag{27}$$

Global existence and decay of solutions

Then by (18), (25) and (27), we get

$$\int_{t_1}^{t_2} E(t)\,dt \leq \frac{1}{2} C D^2(t) + C_3 \Bigg[C_1 D(t) \sup_{t_1 \leq s \leq t_2} E^{\frac{1}{m+2}}(s) + D^2(t) + C C_* \left(\frac{(p+1)(m+2)}{p-m-1} E(0) \right)^{\frac{1}{m+2}} \sup_{t_1 \leq s \leq t_2} E^{\frac{1}{m+2}}(s) D(t) + C_* \left(\frac{(p+1)(m+2)}{p-m-1} E(0) \right)^{\frac{1}{m+2}} \sup_{t_1 \leq s \leq t_2} E^{\frac{1}{m+2}}(s) D^q(t) \Bigg] \quad (28)$$

Global existence and decay of solutions

By integrating (8) over $[t, t_2]$, we obtain

$$E(t) = E(t_2) + \int_t^{t_2} \left(\|u_\tau\|_{q+1}^{q+1} + \|\nabla u_\tau\|^2 \right) d\tau. \tag{29}$$

Therefore, since $t_2 - t_1 \geq \frac{1}{2}$, we conclude that

$$\int_{t_1}^{t_2} E(t)\, dt \geq (t_2 - t_1) E(t_2) \geq \frac{1}{2} E(t_2).$$

That is,

$$E(t_2) \leq 2 \int_{t_1}^{t_2} E(t)\, dt. \tag{30}$$

Consequently, exploiting (15), (28)-(30), and since $t_1, t_2 \in [t, t+1]$, we get

$$\begin{aligned} E(t) &\leq 2 \int_{t_1}^{t_2} E(t)\, dt + \int_t^{t+1} \left(\|u_\tau\|_{q+1}^{q+1} + \|\nabla u_\tau\|^2 \right) d\tau \\ &= 2 \int_{t_1}^{t_2} E(t)\, dt + D^{q+1}(t). \end{aligned} \tag{31}$$

Global existence and decay of solutions

Then, by (28), we have

$$E(t) \leq \left(\frac{1}{2}C + C_3\right) D^2(t) + D^{q+1}(t) + C_4 \left[D(t) + D^q(t)\right] E^{\frac{1}{m+2}}(t).$$

Hence, we obtain

$$E(t) \leq C_5 \left[D^2(t) + D^{q+1}(t) + D^{\frac{m+2}{m+1}}(t) + D^{\frac{m+2}{m+1}q}(t)\right]. \tag{32}$$

Global existence and decay of solutions

Note that since $E(t)$ is nonincreasing and $E(t) \geq 0$ on $[0, \infty)$,

$$\begin{aligned} D^{q+1}(t) &= E(t) - E(t+1) \\ &\leq E(0). \end{aligned}$$

Thus, we have

$$D(t) \leq E^{\frac{1}{q+1}}(0). \tag{33}$$

It follows from (32) and (33) that

$$\begin{aligned} E(t) &\leq C_5 \left[D^{\frac{m}{m+1}}(t) + D^{q-\frac{1}{m+1}}(t) + 1 + D^{\frac{(m+2)(q-1)}{m+1}}(t) \right] D^{\frac{m+2}{m+1}}(t) \\ &\leq C_5 \left[E^{\frac{m}{(m+1)(q+1)}}(0) + E^{\left(q-\frac{1}{m+1}\right)\frac{1}{q+1}}(0) + 1 + E^{\frac{(m+2)(q-1)}{(m+1)(q+1)}} \right] D^{\frac{m+2}{m+1}}(t) \\ &= C_6 D^{\frac{m+2}{m+1}}(t). \end{aligned}$$

Thus, we get

$$E^{1+\frac{(m+1)q-1}{m+2}}(t) \leq C_7 D^{q+1}(t) \tag{34}$$

Global existence and decay of solutions

Case 1: When $q = 1$ and $m = 0$ from (34), we obtain

$$E(t) \leq C_7 D^2(t) = C_7 \left[E(t) - E(t+1) \right].$$

By Lemma 2, we get

$$E(t) \leq E(0) e^{-w_1 [t-1]^+},$$

where $w_1 = \ln \frac{C_7}{C_7 - 1}$.
Case 2: When $(m+1)q > 1$, applying Lemma 2 to (34) yield

$$E(t) \leq \left(E(0)^{-\alpha} + C_7^{-1} \alpha [t-1]^+ \right)^{-\frac{1}{\alpha}}$$

where $\alpha = \frac{(m+1)q-1}{m+2}$. The proof of Theorem 9 is completed.

Blow up of solutions

In this section, we deal with the blow up of the solution for the problem (1), when $q = 1$. Let us begin by stating the following two lemmas,which will be used later.

Lemma (10)

(Li-Tsai 2003). Let us have $\delta > 0$ *and let* $B(t) \in C^2(0, \infty)$ *be a nonnegative function satisfying*

$$B''(t) - 4(\delta + 1) B'(t) + 4(\delta + 1) B(t) \geq 0. \quad (35)$$

If

$$B'(0) > r_2 B(0) + K_0, \quad (36)$$

with $r_2 = 2(\delta + 1) - 2\sqrt{(\delta + 1)\delta}$, *then* $B'(t) > K_0$ *for* $t > 0$, *where* K_0 *is a constant.*

Blow up of solutions

Lemma (11)

(Li-Tsai 2003). If $H(t)$ is a nonincreasing function on $[t_0, \infty)$ and satisfies the differential inequality

$$\left[H'(t)\right]^2 \geq a + b\left[H(t)\right]^{2+\frac{1}{\delta}}, \text{ for } t \geq t_0, \tag{37}$$

where $a > 0$, $b \in R$, then there exists a finite time T^* such that

$$\lim_{t \longrightarrow T^{*-}} H(t) = 0.$$

Upper bounds for T^* are estimated as follows:

Blow up of solutions

(i) If $b < 0$ and $H(t_0) < \min\left\{1, \sqrt{-\frac{a}{b}}\right\}$ then

$$T^* \leq t_0 + \frac{1}{\sqrt{-b}} \ln \frac{\sqrt{-\frac{a}{b}}}{\sqrt{-\frac{a}{b}} - H(t_0)}.$$

(ii) If $b = 0$, then

$$T^* \leq t_0 + \frac{H(t_0)}{H'(t_0)}.$$

(iii) If $b > 0$, then

$$T^* \leq \frac{H(t_0)}{\sqrt{a}} \text{ or } T^* \leq t_0 + 2^{\frac{3\delta+1}{2\delta}} \frac{\delta c}{\sqrt{a}} \left[1 - \left(1 + cH(t_0)\right)^{-\frac{1}{2\delta}}\right],$$

where $c = \left(\frac{a}{b}\right)^{2+\frac{1}{\delta}}$.

Blow up of solutions

Definition (12)

A solution u of (1) with $q = 1$ is called blow up if there exists a finite time T^* such that

$$\lim_{t \longrightarrow T^{*-}} \left[\int_\Omega u^2 dx + \int_0^t \int_\Omega \left(u^2 + |\nabla u|^2 \right) dx d\tau \right] = \infty. \qquad (38)$$

Let

$$a(t) = \int_\Omega u^2 dx + \int_0^t \int_\Omega \left(u^2 + |\nabla u|^2 \right) dx d\tau, \text{ for } t \geq 0. \qquad (39)$$

Blow up of solutions

Lemma (13)

Assume $m+2 < p+1 < \frac{n(m+2)}{n-(m+2)}$, $m+2 < n$, *and that* $m \leq 4\delta \leq p-1$, *then we have*

$$a''(t) \geq 4(\delta+1)\int_\Omega u_t^2 dx - 4(2\delta+1)E(0) + 4(2\delta+1)\int_0^t \left(\|u_t\|^2 + \|\nabla u_t\|^2 \tag{40}$$

Proof. From (39), we have

$$a'(t) = 2\int_\Omega u u_t dx + \|u\|^2 + \|\nabla u\|^2, \tag{41}$$

$$\begin{aligned} a''(t) &= 2\int_\Omega u_t^2 dx + 2\int_\Omega u u_{tt} dx + 2\int_\Omega (u u_t + \nabla u \nabla u_t)\, dx \\ &= 2\|u_t\|^2 - 2\|\nabla u\|_{m+2}^{m+2} + 2\|u\|_{p+1}^{p+1}. \end{aligned} \tag{42}$$

Blow up of solutions

Lemma (14)

Assume $m+2 < p+1 < \frac{n(m+2)}{n-(m+2)}$, $m+2 < n$ *and one of the following statements are satisfied*
(i) $E(0) < 0$,
(ii) $E(0) = 0$ *and* $\int_\Omega u_0 u_1 dx > 0$,
(iii) $E(0) > 0$ *and*

$$a'(0) > r_2 \left[a(0) + \frac{K_1}{4(\delta+1)} \right] + \|u_0\|^2 \tag{43}$$

holds.
Then $a'(t) > \|u_0\|^2$ *for* $t > t^*$, *where* $t_0 = t^*$ *is given by (44) in case (i) and* $t_0 = 0$ *in cases (ii) and (iii).*
Where K_1 *and* t^* *are defined in (48) and (44), respectively.*

Blow up of solutions

Proof. (i) If $E(0) < 0$, then from (40), we have

$$a'(t) \geq a'(0) - 4(2\delta + 1) E(0) t, \quad t \geq 0.$$

Thus we get $a'(t) > \|u_0\|^2$ for $t > t^*$, where

$$t^* = \max\left\{ \frac{a'(t) - \|u_0\|^2}{4(2\delta + 1) E(0)}, 0 \right\}. \tag{44}$$

(ii) If $E(0) = 0$ and $\int_\Omega u_0 u_1 dx > 0$, then $a''(t) \geq 0$ for $t \geq 0$. We have $a'(t) > \|u_0\|^2$, $t \geq 0$.
(iii) If $E(0) > 0$, we first note that

$$2\int_0^t \int_\Omega u u_t dx d\tau = \|u\|^2 - \|u_0\|^2. \tag{45}$$

Blow up of solutions

By Hölder inequality and Young inequality, we have

$$\|u\|^2 \leq \|u_0\|^2 + \int_0^t \|u\|^2 \, d\tau + \int_0^t \|u_t\|^2 \, d\tau. \tag{46}$$

By Hölder inequality, Young inequality and (46), we have

$$a'(t) \leq a(t) + \|u_0\|^2 + \int_\Omega u_t^2 dx + \int_0^t \|u_t\|^2 \, d\tau. \tag{47}$$

Hence, by (40) and (47), we obtain

$$a''(t) - 4(\delta+1)\, a'(t) + \|u_0\|^2 a(t) + K_1 \geq 0,$$

where

$$K_1 = 4(2\delta+1)\, E(0) + 4(\delta+1)\, \|u_0\|^2. \tag{48}$$

Let

$$b(t) = a(t) + \frac{K_1}{4(\delta+1)}, \quad t > 0.$$

Then $b(t)$ satisfies Lemma 10. Consequently, we get from (43)
$a'(t) > \|u_0\|^2$, $t > 0$, where r_2 is given in Lemma 10

Blow up of solutions

Theorem (15)

Assume $m+2 < p+1 < \frac{n(m+2)}{n-(m+2)}$, $m+2 < n$ *and one of the following statements are satisfied*

(i) $E(0) < 0$,

(ii) $E(0) = 0$ *and* $\int_\Omega u_0 u_1 dx > 0$,

(iii) $0 < E(0) < \frac{\left(a'(t_0) - \|u_0\|^2\right)^2}{8\left[a(t_0) + (T_1 - t_0)\|u_0\|^2\right]}$ *and (43) holds.*

Then the solution u *blow up in finite time* T^* *in the case of (38). In case (i),*

$$T^* \leq t_0 - \frac{H(t_0)}{H'(t_0)}. \tag{49}$$

Blow up of solutions

Furthermore, if $H(t_0) < \min\left\{1, \sqrt{-\frac{a}{b}}\right\}$, we have

$$T^* \leq t_0 + \frac{1}{\sqrt{-b}} \ln \frac{\sqrt{-\frac{a}{b}}}{\sqrt{-\frac{a}{b}} - H(t_0)}, \tag{50}$$

where

$$a = \delta^2 H^{2+\frac{2}{\delta}}(t_0)\left[\left(a'(t_0) - \|u_0\|^2\right)^2 - 8E(0)\, H^{-\frac{1}{\delta}}(t_0)\right] > 0, \tag{51}$$

$$b = 8\delta^2 E(0). \tag{52}$$

In case (ii),

$$T^* \leq t_0 - \frac{H(t_0)}{H'(t_0)}. \tag{53}$$

In case (iii),

$$T^* \leq \frac{H(t_0)}{\sqrt{a}} \text{ or } T^* \leq t_0 + 2^{\frac{3\delta+1}{2\delta}} \left(\frac{a}{b}\right)^{2+\frac{1}{\delta}} \frac{\delta}{\sqrt{a}} \left\{1 - \left[1 + \left(\frac{a}{b}\right)^{2+\frac{1}{\delta}} H(t_0)\right]^{-\frac{1}{2\delta}}\right.$$

Blow up of solutions

Proof. Let

$$H(t)=\left[a(t)+(T_1-t)\|u_0\|^2\right]^{-\delta},\ \text{for } t\in[0,T_1], \tag{55}$$

where $T_1>0$ is a certain constant which will be specified later. Then we get

$$\begin{aligned} H'(t) &= -\delta\left[a(t)+(T_1-t)\|u_0\|^2\right]^{-\delta-1}\left[a'(t)-\|u_0\|^2\right] \\ &= -\delta H^{1+\frac{1}{\delta}}(t)\left[a'(t)-\|u_0\|^2\right], \end{aligned} \tag{56}$$

$$\begin{aligned} H''(t) &= -\delta H^{1+\frac{2}{\delta}}(t)\,a''(t)\left[a(t)+(T_1-t)\|u_0\|^2\right] \\ &\quad +\delta H^{1+\frac{2}{\delta}}(t)(1+\delta)\left[a'(t)-\|u_0\|^2\right]^2. \end{aligned} \tag{57}$$

and

$$H''(t)=-\delta H^{1+\frac{2}{\delta}}(t)\,V(t), \tag{58}$$

where

Blow up of solutions

For simplicity of calculation, we define

$$P_u = \int_\Omega u^2 dx, \qquad R_u = \int_\Omega u_t^2 dx$$
$$Q_u = \int_0^t \|u\|^2 dt, \quad S_u = \int_0^t \|u_t\|^2 dt.$$

From (41), (45) and Hölder inequality, we get

$$\begin{aligned} a'(t) &= 2\int_\Omega u u_t dx + \|u_0\|^2 + 2\int_0^t \int_\Omega u u_t dx dt \\ &\leq 2\left(\sqrt{R_u P_u} + \sqrt{Q_u S_u}\right) + \|u_0\|^2 . \end{aligned} \tag{60}$$

If case (i) or (ii) holds, by (40) we have

$$a''(t) \geq (-4 - 8\delta) E(0) + 4(1+\delta)(R_u + S_u). \tag{61}$$

Blow up of solutions

Thus, from (59)-(61) and (55), we obtain

$$\begin{aligned} V(t) \geq & \left[(-4-8\delta)E(0)+4(1+\delta)(R_u+S_u)\right]H^{-\frac{1}{\delta}}(t) \\ & -4(1+\delta)\left(\sqrt{R_uP_u}+\sqrt{Q_uS_u}\right)^2. \end{aligned}$$

From (39),

$$a(t)=\int_\Omega u^2dx+\int_0^t\int_\Omega u^2dxds=P_u$$

and (55), we get

$$V(t)\geq(-4-8\delta)E(0)H^{-\frac{1}{\delta}}(t)+4(1+\delta)\left[(R_u+S_u)(T_1-t)\|u_0\|^2+\Theta(t$$

where

$$\Theta(t)=(R_u+S_u)(P_u+Q_u)-\left(\sqrt{R_uP_u}+\sqrt{Q_uS_u}\right)^2.$$

Blow up of solutions

By the Schwarz inequality, and $\Theta(t)$ being nonnegative, we have

$$V(t) \geq (-4-8\delta) E(0) H^{-\frac{1}{\delta}}(t), \ t \geq t_0. \quad (62)$$

Therefore, by (58) and (62), we get

$$H''(t) \leq 4\delta(1+2\delta) E(0) H^{1+\frac{1}{\delta}}(t), \ t \geq t_0. \quad (63)$$

By Lemma 13, we know that $H'(t) < 0$ for $t \geq t_0$. Multiplying (63) by $H'(t)$ and integrating it from t_0 to t, we get

$$H'^2(t) \geq a + bH^{2+\frac{1}{\delta}}(t)$$

for $t \geq t_0$, where a, b are defined in (51) and (52) respectively.

Blow up of solutions

By Lemma 13, we know that $H'(t) < 0$ for $t \geq t_0$. Multiplying (63) by $H'(t)$ and integrating it from t_0 to t, we get

$$H'^2(t) \geq a + bH^{2+\frac{1}{\delta}}(t)$$

for $t \geq t_0$, where a, b are defined in (51) and (52) respectively.
If case (iii) holds, by the steps of case (i), we get $a > 0$ if and only if

$$E(0) < \frac{\left(a'(t_0) - \|u_0\|^2\right)^2}{8\left[a(t_0) + (T_1 - t_0)\|u_0\|^2\right]}.$$

Then by Lemma 11, there exists a finite time T^* such that $\lim_{t \longrightarrow T^{*-}} H(t) = 0$ and upper bound of T^* is estimated according to the sign of $E(0)$. This means that (38) holds.

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