Existence and multiplicity for implicit discretization of Nagumo RDE on unbounded domain via variational methods

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Nagumo RDE

One dimensional reaction-diffusion equation for $u = u(x,t)$, $x \in \mathbb{R}$, $t \in [0,\infty)$,

$$\partial_t u = k\partial_{xx} u + f(u). \qquad \text{(PDE)}$$

Two factors:

diffusion	$k\partial_{xx}u$	...	spatial spread of a substance
reaction	$f(u)$	...	local dynamics, sources

Motivation: biological, chemical, economic, social ... phenomena

Nagumo equation: $f(u) = \lambda u(1-u^2)$

$\lambda \geq 0 \quad \rightarrow \quad$ bistable case

$\lambda < 0 \quad \rightarrow \quad$ monostable case

Discrete version

Discretization of (PDE) via finite differences:

- **Spatial variable:**

$$x \in \mathbb{Z}: \quad \partial^2_{xx} u(x,t) \quad \sim \quad \Delta^2_{xx} u(x-1,t+h)$$
$$= u(x-1,t+h) - 2u(x,t+h) + u(x+1,t+h)$$

- **Time variable:**

$$t \in h\mathbb{N}_0: \quad \partial_t u(x,t) \quad \sim \quad \Delta_t u(x,t) = \frac{u(x,t+h) - u(x,t)}{h}$$

Discretization of the right-hand side of (PDE):

explicit (at time t)	$\rightarrow$	existence and uniqueness is **simple**
implicit (at time $t+h$)	$\rightarrow$	existence and uniqueness **???**

Problem

Consider the following implicit discrete Nagumo equation

$$\begin{cases} \Delta_t u(x,t) = k\Delta_{xx}^2 u(x-1,t+h) + \lambda u(x,t+h)\left(1-u^2(x,t+h)\right), \\ u(x,0) = \varphi(x), \end{cases} \tag{E}$$

with:

- $x \in \mathbb{Z}$
- $t \in h\mathbb{N}_0$, $h > 0$
- $\lambda \in \mathbb{R}$
- $\varphi : \mathbb{Z} \to \mathbb{R}$

Example - infinitely many solutions

- let $\lambda = 0$ (no reaction) and $\varphi(x) = 0$ for all $x \in \mathbb{Z}$ in (E)
- for $t = h$ we obtain the following second order difference equation without initial conditions

$$u(x+1) + \frac{1-2h}{h} u(x) + u(x-1) = 0, \quad x \in \mathbb{Z}$$

- one can obtain (using the theory of IVPs for difference equations) that, e.g., for $h < \frac{1}{4}$

$$u(x) = \frac{u(1) - \lambda_2 u(0)}{\lambda_1 - \lambda_2} \lambda_1^x + \frac{u(1) - \lambda_1 u(0)}{\lambda_2 - \lambda_1} \lambda_2^x$$

with

$$\lambda_{1,2} = \frac{1 - 2h \pm \sqrt{1-4h}}{2h}, \quad \text{i.e.,} \quad \lambda_1 > 1, \quad |\lambda_2| < \lambda_1$$

- if we set for example $u(1) = a \in [0, \infty)$ and $u(0) = 0$, then:
  1. $u(x) \to \infty$ provided $a > 0$
  2. $u(x) \equiv 0$ provided $a = 0$
- there exist infinitely many solutions of (E) at $t = h$
- there are all unbounded except the vanishing one

Bounded ℓ^2 solutions

- we restrict ourselves to locally bounded solutions, i.e., $\{u(x,t)\}_{x\in\mathbb{Z}} = u(\cdot,t)$ bounded for every fixed $t \in h\mathbb{N}_0$
- we want to study the variational structure of corresponding energy functionals
- let $\{\varphi(x)\}_{x\in\mathbb{Z}} = \varphi \in \ell^2(\mathbb{Z})$ and prove the existence of solution for which there is

$$\{u(x,t)\}_{x\in\mathbb{Z}} = u(\cdot,t) \in \ell^2(\mathbb{Z}) \quad \text{for all fixed} \quad t \in h\mathbb{N}_0$$

Fixed point problem

- define operators $L, N : \ell^2 \to \ell^2$:

$$(Lu)_i := ku_{i-1} - 2ku_i + ku_{i+1}, \quad i \in \mathbb{Z}$$

$$(N(u))_i = u_i \left(1 - u_i^2\right), \quad i \in \mathbb{Z}$$

- $L \in \mathcal{L}(\ell^2)$ is negative self-adjoint and N is continuous
- (E) is equivalent to the abstract difference equation on ℓ^2

$$\begin{cases} \frac{1}{h}\left(u(\cdot, t+h) - u(\cdot, t)\right) = L(u(\cdot, t+h)) + \lambda N(u(\cdot, t+h)), \\ u(\cdot, 0) = \varphi \end{cases}$$

- let $t \in h\mathbb{N}_0$ be fixed and $u(\cdot, t) \in \ell^2$ known, denoting

$$b = u(\cdot, t) \in \ell^2(\mathbb{Z}), \quad u = u(\cdot, t+h) \in \ell^2$$

we obtain the fixed point problem in ℓ^2

$$u = b + hL(u) + h\lambda N(u) \qquad \text{(FP)}$$

Variational formulation

- the **energy functional** for (FP) is given by

$$\mathcal{F}(u) = \frac{1 - h\lambda}{2}\|u\|_2^2 - (b, u)_2 - \frac{h}{2}(Lu, u)_2 + \frac{h\lambda}{4}\|u\|_4^4$$

Lemma

$\tilde{u} \in \ell^2$ is a critical point of $\mathcal{F}$ if and only if $\tilde{u}$ is the solution of (FP).

- $\mathcal{F} \in C^1(\ell^2, \mathbb{R})$
- there is

$$(\nabla\mathcal{F}(u), w)_2 = (u - b - hL(u) - h\lambda N(u), w)_2 \, .$$

Bistable case $\lambda \geq 0$

Theorem

Let $\lambda \geq 0$ and assume $h(\lambda + 4k) < 1$ and $\varphi \in \ell^2$. Then the problem (E) *has a unique solution $u(x,t)$ that exists for all $x \in \mathbb{Z}$, $t \in h\mathbb{N}_0$ and satisfies*

$$\|u(\cdot,t)\|_2 < \infty \quad \textit{for all} \quad t \in h\mathbb{N}_0.$$

- $\mathcal{F}$ is **globally convex** and **weakly coercive** on ℓ^2
- $\mathcal{F}$ has a global minimizer $\Rightarrow$ local solution
- mathematical induction

Case $\lambda < 0$

The geometry of $\mathcal{F}$ changes!

Theorem

Let $\lambda < 0$ and assume $h(\lambda + 4k) < 1$ and $u(x,t)$ is a solution of (E) *at a fixed time $t \in h\mathbb{N}_0$ such that $\|u(\cdot,t)\|_2$ is "sufficiently small". Then there exists a solution $u(\cdot, t+h)$ of the problem* (E) *at time $t+h$ such that $\|u(x,t+h)\|_2 < \infty$.*

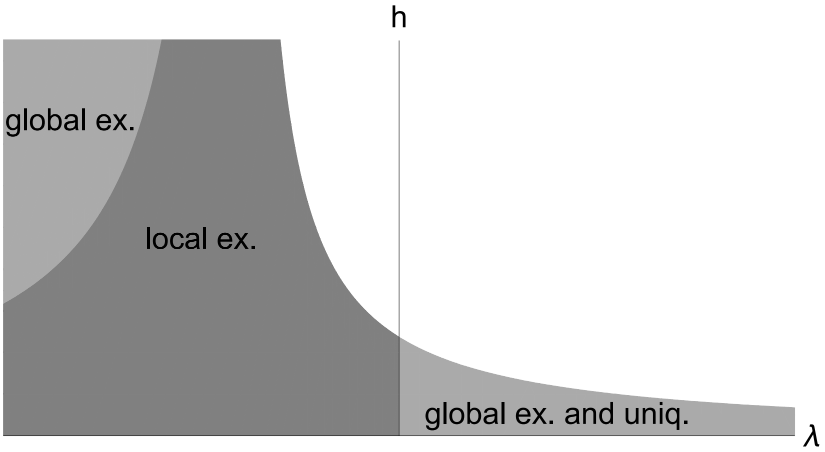
- $\mathcal{F}$ **locally convex** on a ball $\overline{B}(o,R)$
- $\mathcal{F}$ has a local minimizer
- only local solution at $t+h$

Theorem

Let $\lambda < 0$ and assume $h(\lambda + 4k) \leq -2$ and $\|\varphi\|_2$ "sufficiently small". Then the problem (E) *has a solution $u(x,t)$ that exists for all $x \in \mathbb{Z}$, $t \in h\mathbb{N}_0$.*

- more restrictive assumptions on parameters
- mathematical induction and $\|u(\cdot, t+h)\|_2$ also "sufficiently small" in the induction step

Summarizing figure



Case $\lambda < 0$ and mountain pass geometry

Mountain pass theorem (Ambrosetti, Rabinowitz):

Let X be a real Banach space and $\mathcal{F} \in C^1(X, \mathbb{R})$ satisfy:

- there exists $e \in X$ and $\rho > 0$ such that $\|e\| > \rho$ and

$$\inf_{\|u\|=\rho} \mathcal{F}(u) > \mathcal{F}(o) \geq \mathcal{F}(e), \tag{MP}$$

- the Palais-Smale condition: "Any sequence $\{u^n\} \subset X$ such that

$$\mathcal{F}(u^n) \to c \in \mathbb{R} \quad \text{and} \quad \nabla \mathcal{F}(u^n) \to o \in X \tag{PS-A}$$

  has a convergent subsequence."

Then $c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \mathcal{F}(\gamma(t))$ where
$\Gamma := \{\gamma \in C([0,1], X) : \gamma(0) = o, \gamma(1) = e\}$ is a critical value of $\mathcal{F}$.

Lemma

Let $\lambda < 0$ and assume $h(\lambda + 4k) < 1$ and $\|b\|_2$ be "sufficiently small". Then there exist $e \in \ell^2$ and $\rho > 0$ such that $\|e\|_2 > \rho$ and $\mathcal{F}$ satisfies (MP).

Palais-Smale condition - boundedness

Structure of proof

- every $\{u^n\}_{n\in\mathbb{N}} \subset \ell^2$ satisfying (PS-A) contains a bounded subsequence
- pass to a weakly convergent subsequence $u^n \rightharpoonup u$ and show that it converges strongly as well

Lemma

Let $\lambda < 0$, $h > 0$, $h(\lambda + 4k) < 1$, $b \in \ell^2$ and $\mathcal{F}$ be the energy functional. Then every sequence $\{u^n\} \subset \ell^2$ satisfying (PS-A) *is bounded.*

- from (PS-A) one can obtain that a Palais-Smale sequence satisfies for a.a. $n \in \mathbb{N}$

$$K + L\|u^n\|_2 \geq M\|u^n\|_2^2, \quad K, L, M > 0$$

(PS) condition - convergence

- pass to a subsequence $u^n \rightharpoonup u$
- typical mountain pass argument works with

$$(\nabla\mathcal{F}(u^n) - \nabla\mathcal{F}(u), u^n - u)_2 \to 0.$$

For our energy functional $\mathcal{F}$ we obtain the estimate

$$(1 - h\lambda)\|u^n - u\|_2^2 \leq (\nabla\mathcal{F}(u^n) - \nabla\mathcal{F}(u), u^n - u)_2 \underbrace{-h\lambda \sum_{i\in\mathbb{Z}} \left((u_i^n)^3 - u_i^3\right)(u_i^n - u_i)}_{\text{PROBLEMATIC TERM}}.$$

Case $\lambda < 0$ - conjectures

We have tried:

- use the boundedness of $\{u^n\}_{n\in\mathbb{N}}$

$$\underbrace{(1 - h\lambda + h\lambda K(h))}_{\text{it has not to be nonnegative}} \|u^n - u\|_2^2 \leq (\nabla\mathcal{F}(u^n) - \nabla\mathcal{F}(u), u^n - u)_2$$

- pass with the limit into the sum in the "problematic term"

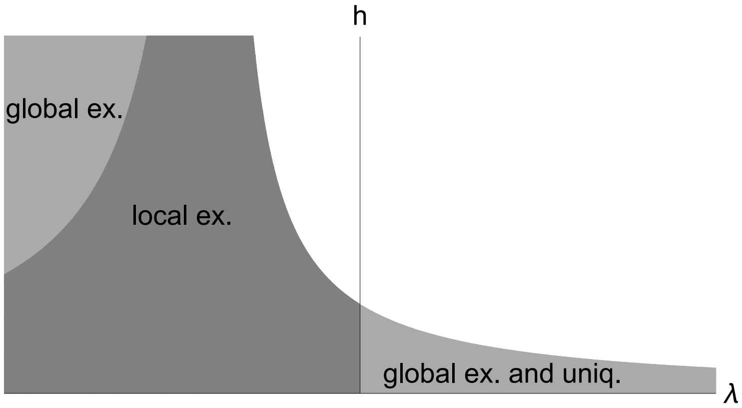
Conjecture

Let $\lambda < 0$ and assume $h(\lambda + 4k) < 1$ and $\|b\|_2$ "sufficiently small". Then the functional $\mathcal{F}$ has at least two critical points.

Conjecture

Let $\lambda < 0$, $h(\lambda + 4k) < 1$, $h > 0$ and $u(x,t)$ be a solution of (E) at a fixed time $t \in h\mathbb{N}_0$ such that $\|u(\cdot,t)\|_2$ is "sufficiently small". Then the problem (E) has at least two solutions $u_1(x,t+h)$, $u_2(x,t+h)$ at time $t+h$ such that $u_j(\cdot,t+h) \in \ell^2$, $j = 1,2$.

Summary, open questions



λ	$\lambda < 0$ $(-\infty, -\frac{2}{h} - 4k]$	$(-\frac{2}{h} - 4k, \frac{1}{h} - 4k)$	$\lambda \geq 0$ $[0, \frac{1}{h} - 4k)$	$[\frac{1}{h} - 4k, \infty)$
Geometry of $\mathcal{F}$	mountain pass		globally convex	?
Existence	global	local	global	?
Uniqueness in ℓ^2	?	?	yes	?

References

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Thank you.