Existence results for discrete (p, q)-Laplacian equations

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Introduction

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Some references

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A. Cabada, A. Iannizzotto, S. Tersian,
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The Problem

Let $N \in \mathbb{Z}_+$, $[1, N] := \{1, \ldots, N\}$, $1 < q < p < +\infty$, $\lambda \in]0, +\infty[$.

$$\begin{cases} -\Delta_p u(z-1) - \Delta_q u(z-1) + \alpha(z)\phi_p(u(z)) + \beta(z)\phi_q(u(z)) = \lambda g(z, u(z)), \\ \hfill \text{for all } z \in [1, N], \\ u(0) = u(N+1) = 0, \end{cases}$$

- $\Delta u(z-1) = u(z) - u(z-1)$ is the forward difference operator,
- $\Delta_p u(z-1) := \Delta(\phi_p(\Delta u(z-1))) = \phi_p(\Delta u(z)) - \phi_p(\Delta u(z-1))$ is the discrete p-Laplacian,
- $\phi_p : \mathbb{R} \to \mathbb{R}$ is given as $\phi_p(u) = |u|^{p-2}u$ with $u \in \mathbb{R}$,
- $\alpha, \beta : [1, N+1] \to \mathbb{R}$,
- $g : [1, N+1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function with $g(N+1, t) = 0$ for all $t \in \mathbb{R}$.

The Problem

Let $N \in \mathbb{Z}_+$, $[1, N] := \{1, \ldots, N\}$, $1 < q < p < +\infty$, $\lambda \in]0, +\infty[$.

$$\begin{cases} -\Delta_p u(z-1) - \Delta_q u(z-1) + \alpha(z)\phi_p(u(z)) + \beta(z)\phi_q(u(z)) = \lambda g(z, u(z)), \\ \hfill \text{for all } z \in [1, N], \\ u(0) = u(N+1) = 0. \end{cases}$$

We consider the following hypotheses:

(h_1) $g(z, 0) \geq 0$ for all $z \in [1, N]$, and $g(z, t) = g(z, 0)$ for all $t < 0$ and for all $z \in [1, N]$;

(h_2) $\alpha(z), \beta(z) \geq 0$ for all $z \in [1, N]$.

The features

- One can obtain existence results under more general assumptions (on the nonlinearity) than those required for continuous differential problems;
- the settings enable us to work with practical (discrete) cases, arising in numerical analysis as discretized versions of continuous operators;
- numerical simulations play a key-role in evaluating theoretical results, to suggest or disprove theoretical assumptions (i.e., suitable directions of research);
- one does not need the Ambrosetti-Rabinowitz condition $(\exists\, \theta > p,\ s_0 > 0 \,:\, s\, g(z,s) \geq \theta G(z,s) > 0 \text{ for } |s| \geq s_0)$;
- more general assumptions than the sublinearity at zero and the superlinearity at infinity can be used.

Mathematical background

By X and X^* we mean a Banach space and its topological dual, respectively. We consider the N-dimensional Banach space

$$X_d = \{u : [0, N+1] \to \mathbb{R} \quad \text{such that} \quad u(0) = u(N+1) = 0\},$$

and define the norm

$$\|u\|_{r,h} := \left(\sum_{z=1}^{N+1} \left[|\Delta u(z-1)|^r + h(z) \, |u(z)|^r \right] \right)^{\frac{1}{r}},$$

with $h : [0, N+1] \to [0, +\infty[$ and $r \in]1, +\infty[$.
We have (see [6]) the inequality

$$\|u\|_\infty \leq \frac{(N+1)^{\frac{r-1}{r}}}{2} \|u\|_{r,h} \quad \text{for all } u \in X_d, \tag{1}$$

where $\|u\|_\infty := \max_{z \in [1,N]} |u(z)|$ is the usual sup-norm.

Mathematical background

Proposition 1

Let $h = \sum_{z=1}^{N} h(z)$. The following inequalities hold

$$\frac{2}{N+1}\|u\|_\infty \le \|u\|_{r,h} \le (2^r N + h)^{\frac{1}{r}}\|u\|_\infty .$$

Proof. The left inequality follows by (1). Since

$$\begin{aligned}\|u\|_{r,h}^r &= \sum_{z=1}^{N+1} [|\Delta u(z-1)|^r + h(z)|u(z)|^r] \\ &\le 2\|u\|_\infty^r + \sum_{z=2}^{N} 2^r\|u\|_\infty^r + \|u\|_\infty^r \sum_{z=1}^{N} h(z) \\ &= [2^r(N-1) + 2 + h]\|u\|_\infty^r \le [2^r N + h]\|u\|_\infty^r ,\end{aligned}$$

we deduce easily the right inequality.

Mathematical background

Let X_d be endowed with the norm

$$\|u\| = \|u\|_{p,\alpha} + \|u\|_{q,\beta},$$

where α and β are the coefficients of ϕ_p and ϕ_q in (P_d).
We consider the function $G : [1, N+1] \times \mathbb{R} \to \mathbb{R}$ given as

$$G(z,t) = \int_0^t g(z,\xi)d\xi, \quad \text{for all } t \in \mathbb{R},\ z \in [1, N+1],$$

and the functional $B : X_d \to \mathbb{R}$ given as

$$B(u) = \sum_{z=1}^{N+1} G(z, u(z)), \quad \text{for all } u \in X_d.$$

It is clear that $B \in C^1(X_d, \mathbb{R})$ and

$$\langle B'(u), v\rangle = \sum_{z=1}^{N+1} g(z, u(z))v(z), \quad \text{for all } u, v \in X_d.$$

Mathematical background

Consider the functionals $A_1, A_2 : X_d \to \mathbb{R}$ given as

$$A_1(u) = \frac{1}{p}\|u\|_{p,\alpha}^p \quad \text{and} \quad A_2(u) = \frac{1}{q}\|u\|_{q,\beta}^q, \quad \text{for all } u \in X_d.$$

Obviously, $A_1, A_2 \in C^1(X_d, \mathbb{R})$ and we have the Gâteaux derivatives at $u \in X_d$:

$$\langle A_1'(u), v\rangle = \sum_{z=1}^{N+1} \phi_p(\Delta u(z-1))\Delta v(z-1) + \alpha(z)\phi_p(u(z))v(z),$$

$$\langle A_2'(u), v\rangle = \sum_{z=1}^{N+1} \phi_q(\Delta u(z-1))\Delta v(z-1) + \beta(z)\phi_q(u(z))v(z),$$

for all $u, v \in X_d$.

Mathematical background

For $r \in]1, +\infty[$, we have

$$\begin{aligned}
&\sum_{z=1}^{N+1} \phi_r(\Delta u(z-1))\Delta v(z-1) \\
&= \sum_{z=1}^{N+1} [\phi_r(\Delta u(z-1))v(z) - \phi_r(\Delta u(z-1))v(z-1)] \\
&= \sum_{z=1}^{N+1} \phi_r(\Delta u(z-1))v(z) - \sum_{z=1}^{N} \phi_r(\Delta u(z))v(z) \\
&= -\sum_{z=1}^{N+1} \Delta\phi_r(\Delta u(z-1))v(z).
\end{aligned}$$

Mathematical background

So,

$$\langle A_1'(u), v\rangle = \sum_{z=1}^{N+1}[-\Delta\phi_p(\Delta u(z-1)) + \alpha(z)\phi_p(u(z))]v(z),$$
$$\langle A_2'(u), v\rangle = \sum_{z=1}^{N+1}[-\Delta\phi_q(\Delta u(z-1)) + \beta(z)\phi_q(u(z))]v(z),$$

for all $u, v \in X_d$.

Mathematical background

Let $I_\lambda : X_d \to \mathbb{R}$ be the functional defined by

$$I_\lambda(u) = A_1(u) + A_2(u) - \lambda B(u), \quad \text{for all } u \in X_d.$$

Clearly $I_\lambda(0) = 0$. Also, we get

$$\langle I'_\lambda(u), v\rangle = \sum_{z=1}^{N+1} [-\Delta\phi_p(\Delta u(z-1)) - \Delta\phi_q(\Delta u(z-1)) + \alpha(z)\phi_p(u(z)) + \beta(z)\phi_q(u(z)) - \lambda g(z, u(z))]v(z),$$

for all $u, v \in X_d$.

$u \in X_d$ is a solution of (P_d) iff u is a critical point of I_λ.

Mathematical background

As in

- G. D'Aguì, J. Mawhin, A. Sciammetta, Positive solutions for a discrete two point nonlinear boundary value problem with p-Laplacian, J. Math. Anal. Appl., 447 (2017), 383–397.

our key-theorem is a two positive critical points result of

- G. Bonanno, G. D'Aguì, *Two non-zero solutions for elliptic Dirichlet problems*, Z. Anal. Anwend., 35 (2016), 449–464.

which we arrange according to our notation and further use.

Mathematical background

Theorem 1

Let $X_d = \{u : [0, N+1] \to \mathbb{R}$ *such that* $u(0) = u(N+1) = 0\}$ *and* $A_1, A_2, B \in C^1(X_d, \mathbb{R})$ *three functionals such that* $\inf_{u \in X_d}(A_1(u) + A_2(u)) = A_1(0) + A_2(0) = B(0) = 0$. *Assume that*

(i) *there are* $s \in \mathbb{R}$ *and* $\widehat{u} \in X_d$, *with* $0 < A_1(\widehat{u}) + A_2(\widehat{u}) < s$, *such that*

$$\frac{B(\widehat{u})}{A_1(\widehat{u}) + A_2(\widehat{u})} > \frac{\sup_{u \in (A_1+A_2)^{-1}(]-\infty, s])} B(u)}{s};$$

(ii) $\cdots$

Then I_λ *admits two non-zero critical points* $u_{\lambda,1}, u_{\lambda,2} \in X_d$ *such that* $I_\lambda(u_{\lambda,1}) < 0 < I_\lambda(u_{\lambda,2})$, *for all* $\lambda \in \overline{\Lambda}$.

Mathematical background

Theorem 2

Let $X_d = \{u : [0, N+1] \to \mathbb{R}$ *such that* $u(0) = u(N+1) = 0\}$ *and* $A_1, A_2, B \in C^1(X_d, \mathbb{R})$ *three functionals such that* $\inf_{u \in X_d}(A_1(u) + A_2(u)) = A_1(0) + A_2(0) = B(0) = 0$. *Assume that*

(i) $\cdots$

(ii) *the functional* $I_\lambda : X_d \to \mathbb{R}$ *given as* $I_\lambda(u) = A_1(u) + A_2(u) - \lambda B(u)$ *for all* $u \in X_d$ *satisfies the* (PS)*-condition and it is unbounded from below for all*

$$\lambda \in \overline{\Lambda} := \left] \frac{A_1(\widehat{u}) + A_2(\widehat{u})}{B(\widehat{u})}, \frac{s}{\sup_{u \in (A_1+A_2)^{-1}(]-\infty, s])} B(u)} \right[.$$

Then I_λ *admits two non-zero critical points* $u_{\lambda,1}, u_{\lambda,2} \in X_d$ *such that* $I_\lambda(u_{\lambda,1}) < 0 < I_\lambda(u_{\lambda,2})$, *for all* $\lambda \in \overline{\Lambda}$.

Mathematical background

We recall the Palais-Smale condition.

Definition 3

Let X be a real Banach space and X^* its topological dual. Then, $I_\lambda : X \to \mathbb{R}$ satisfies the Palais-Smale condition if any sequence $\{u_n\}$ such that

(i) $\{I_\lambda(u_n)\}$ is bounded;

(ii) $\lim_{n\to+\infty} \|I'_\lambda(u_n)\|_{X^*} = 0$,

has a convergent subsequence.

Mathematical background

We characterize the functional I_λ as follows.

Lemma 4

Let $m_\infty(z) := \liminf_{t\to+\infty} \frac{G(z,t)}{t^p}$ *and* $m_\infty := \min_{z\in[1,N]} m_\infty(z)$. *If* $m_\infty > 0$, *and* (h_1)-(h_2) *hold true, then* I_λ *satisfies the* (PS)*-condition and it is unbounded from below for all* $\lambda \in \Lambda :=]\frac{(2^p+2^q)N+\alpha+\beta}{q\, m_\infty}, +\infty[$, *where* $\alpha = \sum_{z=1}^N \alpha(z)$ *and* $\beta = \sum_{z=1}^N \beta(z)$.

Proof. As $m_\infty > 0$, let $\lambda > \frac{(2^p+2^q)N+\alpha+\beta}{q\, m_\infty}$ and $m \in \mathbb{R}$ such that $m_\infty > m > \frac{(2^p+2^q)N+\alpha+\beta}{q\lambda}$. We consider a sequence $\{u_n\} \subset X_d$ such that $I_\lambda(u_n) \to c \in \mathbb{R}$ and $I'_\lambda(u_n) \to 0$ in X_d^*, as $n \to +\infty$. Let $u_n^+ = \max\{u_n, 0\}$ and $u_n^- = \max\{-u_n, 0\}$ for all $n \in \mathbb{N}$.

Mathematical background

We show that the sequence $\{u_n^-\}$ is bounded. We get

$$\begin{aligned}|\Delta u_n^-(z-1)|^p &\leq |\Delta u_n^-(z-1)|^{p-2}\Delta u_n^-(z-1)\Delta u_n^-(z-1)\\ &\leq -|\Delta u_n(z-1)|^{p-2}\Delta u_n(z-1)\Delta u_n^-(z-1)\\ &= -\phi_p(\Delta u_n(z-1))\Delta u_n^-(z-1),\end{aligned}$$

for all $z \in [1, N+1]$. Moreover,

$$\alpha(z)|u_n^-(z)|^p = -\alpha(z)|u_n(z)|^{p-2}u_n(z)u_n^-(z) = -\alpha(z)\phi_p(u_n(z))u_n^-(z),$$

for all $z \in [1, N+1]$. So,

$$\|u_n^-\|_{p,\alpha}^p = \sum_{z=1}^{N+1}[|\Delta u_n^-(z-1)|^p + \alpha(z)|u_n^-(z)|^p]$$

$$\leq -\sum_{z=1}^{N+1}[\phi_p(\Delta u_n(z-1))\Delta u_n^-(z-1) + \alpha(z)\phi_p(u_n(z))u_n^-(z)]$$

$$= -\langle A_1'(u_n), u_n^-\rangle.$$

Mathematical background

Analogously, we get $\|u_n^-\|_{q,\beta}^q \leq -\langle A_2'(u_n), u_n^-\rangle$. On the other hand, one has

$$\langle B'(u_n), u_n^-\rangle = \sum_{z=1}^{N+1} g(z, u_n(z))u_n^-(z) \geq 0 \quad \text{(by } (h_1)).$$

So,

$$\|u_n^-\|_{p,\alpha}^p \leq \|u_n^-\|_{p,\alpha}^p + \|u_n^-\|_{q,\beta}^q$$
$$\leq -\langle A_1'(u_n), u_n^-\rangle - \langle A_2'(u_n), u_n^-\rangle + \lambda\langle B'(u_n), u_n^-\rangle = -\langle I_\lambda'(u_n), u_n^-\rangle,$$

for all $n \in \mathbb{N}$, which leads to $\|u_n^-\|_{p,\alpha}^{p-1} \to 0$ as $n \to +\infty$. Similarly, we deduce that $\|u_n^-\|_{q,\beta}^{q-1} \to 0$ as $n \to +\infty$, and so $\|u_n^-\| \to 0$ as $n \to +\infty$. We deduce that there is $\rho > 0$ such that

$$\|u_n^-\| \leq \rho \quad \Rightarrow \quad \|u_n^-\|_\infty \leq \frac{\rho + \rho N}{2} := \gamma, \quad \text{for all } n \in \mathbb{N}.$$

Mathematical background

We assume that $\{u_n\}$ is unbounded, which means that $\{u_n^+\}$ is unbounded. We may suppose that $\|u_n\| \to +\infty$ as $n \to +\infty$. By the assumption on m_∞, we deduce that

there is $\delta_z \geq \max\{\gamma, 1\}$ such that $G(z,t) > mt^p$ for all $t > \delta_z$.

For all $z \in [1, N]$, as $G(z, \cdot)$ is a continuous function, there is

$C(z) \geq 0$ such that $G(z,t) \geq m|t|^p - C(z)$ for all $t \in [-\gamma, \delta_z]$.

$$\Rightarrow G(z,t) \geq m|t|^p - C(z) \text{ for all } t \geq -\gamma, \text{ all } z \in [1, N].$$

It follows

$$B(u_n) = \sum_{z=1}^{N+1} G(z, u_n(z)) \geq \sum_{z=1}^{N} m|u_n(z)|^p - C \geq m\|u_n\|_\infty^p - C,$$

for all $n \in \mathbb{N}$, where $C = \sum_{z=1}^{N} C(z)$.

Mathematical background

For all u_n such that $\|u_n\|_\infty \geq 1$, we get

$$\begin{aligned} I_\lambda(u_n) &= A_1(u_n) + A_2(u_n) - \lambda B(u_n) = \frac{1}{p}\|u_n\|^p_{p,\alpha} + \frac{1}{q}\|u_n\|^q_{q,\beta} - \lambda B(u_n) \\ &\leq \left(\frac{2^p N + \alpha}{p} + \frac{2^q N + \beta}{q}\right)\|u_n\|^p_\infty - \lambda m\|u_n\|^p_\infty + \lambda C \\ &\leq \left[\frac{(2^p + 2^q)N + \alpha + \beta}{q} - \lambda m\right]\|u_n\|^p_\infty + \lambda C, \end{aligned}$$

for all $n \in \mathbb{N}$. So, since $\frac{(2^p+2^q)N+\alpha+\beta}{q} - \lambda m < 0$, we get

$$I_\lambda(u_n) \to -\infty \text{ as } n \to +\infty \; (\|u_n\| \to +\infty \Rightarrow \|u_n\|_\infty \to +\infty).$$

This is absurd and so $\{u_n\}$ is bounded. So, I_λ satisfies the (PS)-condition.
Again reasoning on a sequence $\{u_n\} \subset X_d$ such that $\{u_n^-\}$ is bounded and $\|u_n\| \to +\infty$ as $n \to +\infty$, we deduce that $I_\lambda(u_n) \to -\infty$ as $n \to +\infty$ and so I_λ is unbounded from below.

Main results

Let $h : [0, N+1] \to [0, +\infty[$ and $r \in]1, +\infty[$. If $-\Delta(\phi_r(\Delta u(z-1))) + h(z)\phi_r(u(z)) \geq 0$ and $u(z) \leq 0$, then

$$\Delta u(z) \begin{cases} \leq 0 & \text{if } \Delta u(z-1) \leq 0; \\ < 0 & \text{if } \Delta u(z-1) < 0. \end{cases} \tag{2}$$

Indeed, if $u(z) \leq 0$ then $\phi_r(u(z)) \leq 0$ and hence $-\Delta(\phi_r(\Delta u(z-1))) \geq 0$. So, we have $\phi_r(\Delta u(z)) \leq \phi_r(\Delta u(z-1))$, which implies that (2) holds true.

Main results

Let $C_+ := \{u \in X_d : u(z) > 0 \text{ for all } z \in [1, N]\}$.

A solution u of the problem (P_d) is positive if $u \in C_+$.

We establish the following result:

Theorem 5

Let $u \in X_d$ be fixed so that one of the following inequalities holds true for each $z \in [1, N]$:

(a) $u(z) > 0$;

(b) $-\Delta(\phi_p(\Delta u(z-1))) + \alpha(z)\phi_p(u(z)) \geq 0$;

(c) $-\Delta(\phi_q(\Delta u(z-1))) + \beta(z)\phi_q(u(z)) \geq 0$.

Then, either $u \in C_+$ or $u \equiv 0$, provided that (h_2) holds too.

Main results

Proof. Let $u \in X_d \setminus \{0\}$ and $J = \{z \in [1, N] : u(z) \leq 0\}$. If $J = \emptyset$, then $u \in C_+$. By absurd, we assume that $J \neq \emptyset$. If $\min J = 1$, then from (2) we deduce that $\Delta u(1) \leq 0$, which implies $u(2) \leq 0$. By iterating this argument, we get easily

$$0 = u(N+1) \leq u(N) \leq \cdots \leq u(2) \leq u(1) \leq 0,$$

which leads to contradiction (i.e., $u \equiv 0$). On the other hand, if $\min J = j \in [2, N]$, then $\Delta u(j-1) = u(j) - u(j-1) < 0$ (note that $u(j-1) > 0$). By (2), we obtain

$$\Delta u(j) < 0 \quad \Rightarrow \quad u(j+1) < u(j) \leq 0.$$

By iterating this argument, we get easily

$$u(N+1) < u(N) < \cdots < u(j+1) < u(j) \leq 0,$$

which leads to contradiction (i.e., $u(N+1) < 0$). Then, $J = \emptyset$ and hence $u \in C_+$.

Main results

Let $\xi^+ = \max\{0, \xi\}$ and denote by $g_+ : [1, N+1] \times \mathbb{R} \to \mathbb{R}$ the function given as $g_+(z, \xi) = g(z, \xi^+)$ for all $z \in [1, N]$, all $\xi \in \mathbb{R}$.

Remark 1

If the function $g : [1, N+1] \times \mathbb{R} \to \mathbb{R}$ is such that $g(z, 0) \geq 0$ for all $z \in [1, N]$, then g_+ satisfies the condition (h_1).

Now, consider the function $G^+ : [1, N+1] \times \mathbb{R} \to \mathbb{R}$ given as

$$G^+(z, t) = \int_0^t g_+(z, \xi) d\xi, \quad \text{for all } t \in \mathbb{R}, \ z \in [1, N+1],$$

and the functional $B^+ : X_d \to \mathbb{R}$ defined by

$$B^+(u) = \sum_{z=1}^{N+1} G^+(z, u(z)), \quad \text{for all } u \in X_d.$$

Main results

It is clear that $B^+ \in C^1(X_d, \mathbb{R})$. Also, the functional $I_\lambda^+ : X_d \to \mathbb{R}$ given as

$$I_\lambda^+(u) = A_1(u) + A_2(u) - \lambda B^+(u), \quad \text{for all } u \in X_d,$$

has as critical points the solutions of the following problem (P_d^+)

$$\begin{cases} -\Delta_p u(z-1) - \Delta_q u(z-1) + \alpha(z)\phi_p(u(z)) \\ \qquad\qquad +\beta(z)\phi_q(u(z)) = \lambda g_+(z, u(z)), \text{ for all } z \in [1, N], \\ u(0) = u(N+1) = 0. \end{cases}$$

Main results

Remark 2

It is immediate to check that Lemma 4 holds true for the functional I_λ^+, if we assume that $g(z,0) \geq 0$ for all $z \in [1,N]$. In fact, this ensures that (h_1) holds for g_+ (by Remark 1).

The proof of the following proposition is an immediate consequence of Theorem 5.

Proposition 2

If the function $g : [1, N+1] \times \mathbb{R} \to \mathbb{R}$ is such that $g(z,0) \geq 0$ for all $z \in [1,N]$, then each non-zero critical point of I_λ^+ is a positive solution of (P_d), provided that (h_2) holds true.

Main results

Proof. We note that each positive solution $u \in X_d$ of (P_d^+) is a positive solution of (P_d), since $g_+(z, u(z)) = g(z, u(z))$ for all $z \in [1, N]$. So, we prove that the non-zero solutions of (P_d^+) are positive. If $u \in X_d \setminus \{0\}$ is a solution of (P_d^+) then, for all $z \in [1, N]$ such that $u(z) \leq 0$, we have

$$\begin{aligned} &-\Delta_p u(z-1) - \Delta_q u(z-1) + \alpha(z)\phi_p(u(z)) + \beta(z)\phi_q(u(z)) \\ &= \lambda g(z, u^+(z)) = \lambda g(z, 0) \geq 0. \end{aligned}$$

This ensures that either (b) or (c) holds for each $z \in [1, N]$ such that $u(z) \leq 0$. So, by an application of Theorem 5, we conclude that $u \in C_+$. It follows that the non-zero solutions of (P_d^+) are positive and hence are positive solutions of (P_d).

Main results

Theorem 6

Let $g : [1, N+1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that $g(z,0) \geq 0$ for all $z \in [1, N]$ and $g(N+1, t) = 0$ for all $t \in \mathbb{R}$. Assume that (h_2) holds true, and there exist $c, d \in]0, +\infty[$ with $c > d$ such that the following inequality is satisfied:

$$c^{-p} \sum_{z=1}^{N+1} \max_{0 \leq \xi \leq c} G(z, \xi) < \frac{(N+1)^{1-p}}{p} \min \left\{ \frac{\sum_{z=1}^{N+1} G(z,d)}{d^p p^{-1}(2+\alpha) + d^q q^{-1}(2+\beta)}, \frac{q\, m_\infty}{(2^p + 2^q)N + \alpha + \beta} \right\}. \tag{3}$$

Then the problem (P_d) has at least two positive solutions, for each $\lambda \in \Lambda^*$ with

$$\Lambda^* = \left] \max \left\{ \frac{d^p p^{-1}(2+\alpha) + d^q q^{-1}(2+\beta)}{\sum_{z=1}^{N+1} G(z,d)}, \frac{(2^p + 2^q)N + \alpha + \beta}{q\, m_\infty} \right\}, \frac{p^{-1}(N+1)^{1-p} c^p}{\sum_{z=1}^{N+1} \max_{0 \leq \xi \leq c} G(z,\xi)} \right[.$$

Main results

Proof. We show that there are $s \in \mathbb{R}$ and $\widehat{u} \in X_d$, with $0 < A_1(\widehat{u}) + A_2(\widehat{u}) < s$, such that

$$\frac{B^+(\widehat{u})}{A_1(\widehat{u}) + A_2(\widehat{u})} > \frac{\sup_{u \in (A_1+A_2)^{-1}(]-\infty, s])} B^+(u)}{s}.$$

Let $s := \dfrac{c^p}{p(N+1)^{p-1}}$. For all $u \in (A_1 + A_2)^{-1}(]-\infty, s])$, we have

$$\begin{aligned}
& \frac{1}{p}\|u\|_{p,\alpha}^p + \frac{1}{q}\|u\|_{q,\beta}^q \leq s, \\
\Rightarrow \quad & \frac{1}{p}\|u\|_{p,\alpha}^p \leq s, \\
\Rightarrow \quad & \|u\|_{p,\alpha} \leq (ps)^{\frac{1}{p}}, \\
\Rightarrow \quad & \|u\|_\infty \leq \frac{(N+1)^{\frac{p-1}{p}}}{2}\|u\|_{p,\alpha} \leq \frac{(N+1)^{\frac{p-1}{p}}}{2}(ps)^{\frac{1}{p}} < c \quad \text{(by (1))}.
\end{aligned}$$

Main results

Since $G^+(z,t) \leq G^+(z,0) = G(z,0)$ for all $t < 0$ and $z \in [1,N]$, we have

$$B^+(u) = \sum_{z=1}^{N+1} G^+(z,u(z)) \leq \sum_{z=1}^{N+1} \max_{0\leq\xi\leq c} G(z,\xi),$$

for all $u \in X_d$ with $u \in (A_1+A_2)^{-1}(]-\infty,s])$, and hence

$$\frac{\sup_{u\in(A_1+A_2)^{-1}(]-\infty,s])} B^+(u)}{s} \leq p(N+1)^{p-1} \frac{\sum_{z=1}^{N+1} \max_{0\leq\xi\leq c} G(z,\xi)}{c^p}. \tag{4}$$

Next, let $\widehat{u} \in X_d$ be given as $\widehat{u}(z) = d$ for all $z \in [1,N]$. We have

Main results

$$A_1(\widehat{u}) + A_2(\widehat{u}) = \frac{(2+\alpha)d^p}{p} + \frac{(2+\beta)d^q}{q}$$
$$= d^p p^{-1}(2+\alpha) + d^q q^{-1}(2+\beta),$$

$$\Rightarrow \quad \frac{B^+(\widehat{u})}{A_1(\widehat{u}) + A_2(\widehat{u})} = \frac{\sum_{z=1}^{N+1} G(z,d)}{d^p p^{-1}(2+\alpha) + d^q q^{-1}(2+\beta)}$$
$$> p(N+1)^{p-1} \frac{\sum_{z=1}^{N+1} \max_{0\le\xi\le c} G(z,\xi)}{c^p},$$

$$\Rightarrow \quad \frac{B^+(\widehat{u})}{A_1(\widehat{u}) + A_2(\widehat{u})} > \frac{\sup_{u\in(A_1+A_2)^{-1}(]-\infty,s])} B^+(u)}{s} \quad \text{(by (4))}.$$

We observe that $0 < d < c$ implies that

$$\sum_{z=1}^{N+1} G(z,d) \le \sum_{z=1}^{N+1} \max_{0\le\xi\le c} G(z,\xi).$$

Main results

So, by (3), we obtain

$$0 < d^p p^{-1}(2+\alpha) + d^q q^{-1}(2+\beta) < \frac{c^p}{p(N+1)^{p-1}}.$$

Also, we have

$$0 < A_1(\widehat{u})+A_2(\widehat{u}) = d^p p^{-1}(2+\alpha)+d^q q^{-1}(2+\beta) < \frac{c^p}{p(N+1)^{p-1}} = s.$$

By an application of Theorem 2, since the functional I_λ^+ satisfies Lemma 4, we conclude that the problem (P_d^+) has at least two non-zero solutions, for each $\lambda \in \Lambda^*$. Finally, Proposition 2 implies that the two solutions are positive and hence they are positive solutions of the problem (P_d).

Main results

Now, we assume that $g : [1, N+1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that $g(z, 0) \geq 0$ for all $z \in [1, N]$, $g(N+1, t) = 0$ for all $t \in \mathbb{R}$, and

$$\limsup_{\xi \to 0^+} \frac{G(z,\xi)}{\xi^p} = +\infty \text{ and } \lim_{\xi \to +\infty} \frac{G(z,\xi)}{\xi^p} = +\infty \tag{5}$$

for all $z \in [1, N]$. Note that the second limit in (5) ensures that $m_\infty = +\infty$. On the other hand, the first limit in (5) ensures that

$$\max_{0 \leq \xi \leq c} G(z,\xi) > 0 \quad \text{for all } z \in [1, N], \text{ all } c > 0.$$

So, we put

$$\bar{\lambda} = \frac{1}{p(N+1)^{p-1}} \sup_{c>0} \frac{c^p}{\sum_{z=1}^{N+1} \max_{0 \leq \xi \leq c} G(z,\xi)} > 0.$$

Main results

It follows that for all $\lambda < \overline{\lambda}$ there exists $c > 0$ such that

$$\lambda < \frac{1}{p(N+1)^{p-1}} \frac{c^p}{\sum_{z=1}^{N+1} \max_{0\leq\xi\leq c} G(z,\xi)} > 0.$$

By the first limit in (5), we obtain that there is $d \in]0, c[$ such that

$$\frac{\sum_{z=1}^{N+1} G(z,d)}{d^p p^{-1}(2+\alpha) + d^q q^{-1}(2+\beta)} > \frac{1}{\lambda}.$$

Consequently

$$c^{-p} \sum_{z=1}^{N+1} \max_{0\leq\xi\leq c} G(z,\xi) < \frac{1}{p(N+1)^{p-1}} \frac{\sum_{z=1}^{N+1} G(z,d)}{d^p p^{-1}(2+\alpha) + d^q q^{-1}(2+\beta)}.$$

Auxiliary results

Corollary 7

Let $g : [1, N+1] \times \mathbb{R} \to \mathbb{R}$ *be a continuous function such that* $g(z, 0) \geq 0$ *for all* $z \in [1, N]$ *and* $g(N+1, t) = 0$ *for all* $t \in \mathbb{R}$. *Also, assume that* $\alpha(z), \beta(z) \geq 0$ *for all* $z \in [1, N]$, *and*

$$\limsup_{\xi \to 0^+} \frac{G(z, \xi)}{\xi^p} = +\infty \text{ and } \lim_{\xi \to +\infty} \frac{G(z, \xi)}{\xi^p} = +\infty,$$

for all $z \in [1, N]$. *Then the problem* (P_d) *has at least two positive solutions, for each* $\lambda \in]0, \overline{\lambda}[$.

Auxiliary results

In the case $c, d \in]0,1]$, we have the following result:

Corollary 8

Let $g : [1, N+1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function with $g(N+1, t) = 0$ for all $t \in \mathbb{R}$. Assume also that (h_1)-(h_2) hold true, and there exist $c, d \in]0,1]$ with $c > d$ be such that the following inequality is satisfied:

$$c^{-q} \sum_{z=1}^{N+1} \max_{0 \leq \xi \leq c} G(z, \xi) < 2^q (N+1)^{1-q} \min \left\{ \frac{\sum_{z=1}^{N+1} G(z, d)}{(4+\alpha+\beta) d^q}, \frac{m_\infty}{(2^p + 2^q) N + \alpha + \beta} \right\}. \quad (6)$$

Then the problem (P_d) has at least two positive solutions, for each $\lambda \in \overline{\Lambda}^*$ with

$$\overline{\Lambda}^* = \left] \max \left\{ \frac{4+\alpha+\beta}{q} \frac{d^q}{\sum_{z=1}^{N+1} G(z, d)}, \frac{(2^p + 2^q) N + \alpha + \beta}{q\, m_\infty} \right\}, \frac{2^q q^{-1} (N+1)^{1-q} c^q}{\sum_{z=1}^{N+1} \max_{0 \leq \xi \leq c} G(z, \xi)} \right[.$$

Auxiliary results

The next result is a particular case of Theorem 6. That is, we deal with the problem:

$$(P_{d,\omega})\begin{cases} -\Delta_p u(z-1) - \Delta_q u(z-1) + \alpha(z)\phi_p(u(z)) \\ \qquad\qquad +\beta(z)\phi_q(u(z)) = \lambda\omega(z)f(u(z)), \text{for all } z \in [1,N], \\ u(0) = u(N+1) = 0, \end{cases}$$

where $\omega : [1, N+1] \to [0, +\infty[$ with $\omega(N+1) = 0$ and $f : \mathbb{R} \to [0, +\infty[$.

Auxiliary results

Let $W = \sum_{z=1}^{N} \omega(z)$ and $F(t) = \int_0^t f(\xi)d\xi$ for all $t \in \mathbb{R}$.

Corollary 9

Let $f : \mathbb{R} \to [0, +\infty[$ be a continuous function. Assume that (h_2) holds true, and that there exist $c, d \in]0, +\infty[$ with $c > d$ such that the following inequality is satisfied:

$$c^{-p} F(c) W < \frac{(N+1)^{1-p}}{p} \min \left\{ \frac{F(d)W}{d^p p^{-1}(2+\alpha) + d^q q^{-1}(2+\beta)}, \frac{q\, m_\infty}{(2^p + 2^q)N + \alpha + \beta} \right\}.$$

Then the problem (P_d) has at least two positive solutions, for each $\lambda \in \Lambda^*$ with

$$\Lambda^* = \left] \max \left\{ \frac{d^p p^{-1}(2+\alpha) + d^q q^{-1}(2+\beta)}{F(d)W}, \frac{(2^p + 2^q)N + \alpha + \beta}{q\, m_\infty} \right\}, \frac{p^{-1}(N+1)^{1-p} c^p}{F(c)W} \right[.$$

Auxiliary results

Proof. Consider the function $g : [1, N+1] \times \mathbb{R} \to \mathbb{R}$ given as

$$g(z,\xi) = \omega(z) f(\xi), \quad \text{for all } z \in [1, N+1], \text{ all } \xi \in \mathbb{R},$$

so that

$$\sum_{z=1}^{N+1} \max_{0 \le \xi \le c} G(z,\xi) = F(c) W.$$

Then, all the assumptions of Theorem 6 hold true and so we conclude that the problem $(P_{d,\omega})$ has at least two positive solutions, for each $\lambda \in \overline{\Lambda}^*$.

THANKS FOR YOUR ATTENTION!!!

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