Kurzweil-Stieltjes integral and second order measure equations

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- $-\infty < a < b < \infty$, X is a Banach space,
- $f\colon [a,b] \to X$ is *regulated* on $[a,b]$, if
  $f(s+) := \lim_{\tau \to s+} f(\tau) \in X$ for $s \in [a,b)$, $f(t-) := \lim_{\tau \to t-} f(\tau) \in X$ for $t \in (a,b]$,
- $\Delta^+ f(s) = f(s+) - f(s)$, $\Delta^- f(t) = f(t) - f(t-)$, $\Delta f(t) = f(t+) - f(t-)$.
- $\mathrm{G} = \mathrm{G}([a,b],X)$ is the space of functions $f\colon [a,b] \to X$ regulated on $[a,b]$.
  (G is a Banach space with respect to the norm $\|f\|_\infty = \sup_{t \in [a,b]} \|f(t)\|$).
  - regulated functions are uniform limits of finite step functions,
  - regulated functions have at most countably many points of discontinuity.
- $\mathrm{BV} = \mathrm{BV}([a,b],X) = \left\{ f\colon [a,b] \to X\colon \operatorname{var}_a^b f < \infty \right\}$ is the space of functions with *bounded variation* on $[a,b]$.

- $\mathcal{G} = \{\delta\colon [a,b] \to (0,1)\}$ are gauges on $[a,b]$.
- $\mathcal{P} = \{P = (D,\xi),\ D = \{a = \alpha_0 < \alpha_1 < \ldots < \alpha_m = b\},\ \xi = \{\xi_1, \ldots, \xi_m\} \in [a,b]^m,\ \xi_j \in [\alpha_{j-1}, \alpha_j]\}$ are tagged divisions of $[a,b]$.
- $P = (D,\xi) \in \mathcal{P}$ is δ-fine if $[\alpha_{j-1}, \alpha_j] \subset (\xi_j - \delta(\xi_j), \xi_j + \delta(\xi_j))$ for all j.
- For $F : [a,b] \to L(X)$, $g : [a,b] \to X$, $P = (D,\xi) \in \mathcal{P}$ define

$$S(F\Delta g, P) = \sum_{j=1}^{m} F(\xi_j)\,[g(\alpha_j) - g(\alpha_{j-1})]\,.$$

Definition

$$I = \int_a^b F\,d[g] \iff \begin{cases} \text{for each } \varepsilon > 0 \text{ there is a gauge } \delta \in \mathcal{G} \text{ such that} \\ \qquad \big|S(F\Delta g, P) - I\big| < \varepsilon \\ \text{for every } \delta - \text{fine tagged division } P. \end{cases}$$

$$\int_c^c F\,d[g] = 0.$$

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- $P = (D,\xi) \in \mathcal{P}$ is δ-fine if $[\alpha_{j-1}, \alpha_j] \subset (\xi_j - \delta(\xi_j), \xi_j + \delta(\xi_j))$ for all j.
- For $F : [a,b] \to L(X)$, $g : [a,b] \to X$, $P = (D,\xi) \in \mathcal{P}$ define

$$S(\Delta F\, g, P) = \sum_{j=1}^{m} [F(\alpha_j) - F(\alpha_{j-1})]\, g(\xi_j)\,.$$

Definition

$$I = \int_a^b d\,[F]\,g \quad \Longleftrightarrow \quad \begin{cases} \text{for each } \varepsilon > 0 \text{ there is a gauge } \delta \in \mathcal{G} \text{ such that} \\ \qquad \Big| S(\Delta F\, g, P) - I \Big| < \varepsilon \\ \text{for every } \delta - \text{fine tagged division } P. \end{cases}$$

$$\int_c^c d\,[F]\,g = 0.$$

- RS $\subset$ KS (including "improper" integrals), LS $\subset$ KS, $X = \mathbb{R} \implies$ KS = PS.
- $F : [a, b] \to L(X)$ and $g : [a, b] \to X$ are regulated $\implies$

  $$\int_a^b F\, d[g] \text{ and } \int_a^b d[F]\, g \quad \text{exist whenever}$$

  one of the functions F, g has a bounded variation.

ASSUME:

- $F,\ F_k \in \mathrm{G}$ for $k \in \mathbb{N}$, $g \in \mathrm{BV}$,
- $F_k \rightrightarrows F$.

THEN: $\displaystyle\int_a^t F_k\, d\,[g] \rightrightarrows \int_a^t F\ \ d\,[g]$ on $[a,b]$.

ASSUME:

- $F \in \mathrm{BV}$, $g, g_k \in \mathrm{G}$ for $n \in \mathbb{N}$,
- $g_k \rightrightarrows g$.

THEN: $\displaystyle\int_a^t F\, d\,[g_k] \rightrightarrows \int_a^t F\, d\,[g]$ on $[a,b]$.

ASSUME:

- $F,\ F_k \in \mathrm{G}$, $g, g_k \in \mathrm{BV}$ for $k \in \mathbb{N}$,
- $F_k \rightrightarrows F$, $g_k \rightrightarrows g$,
- $\alpha^* := \sup\{\mathrm{var}_a^b\, g_k : k \in \mathbb{N}\} < \infty$.

THEN: $\displaystyle\int_a^t F_k\, d\,[g_k] \rightrightarrows \int_a^t F\, d\,[g]$ on $[a,b]$.

BOUNDED CONVERGENCE THEOREM

(i) ASSUME:

- $F \in \mathrm{BV}, \quad g, g_k \in \mathrm{G}$ for $k \in \mathbb{N}$,
- $g_k(t) \to g(t)$ on $[a, b]$,
- $\|g_k\|_\infty \le \gamma^* < \infty$ for $k \in \mathbb{N}$.

THEN: $$\int_a^b d[F]\, g_k \to \int_a^b d[F]\, g.$$

(ii) ASSUME:

- $g \in \mathrm{BV}, \quad F, F_k \in \mathrm{G}$ for $k \in \mathbb{N}$,
- $F_k(t) \to F(t)$ on $[a, b]$,
- $\|F_k\|_\infty \le \varkappa^* < \infty$ for $k \in \mathbb{N}$.

THEN: $$\int_a^b F_k\, d[g] \to \int_a^b F\, d[g].$$

$$\text{(L)} \qquad x(t) = \widetilde{x} + \int_{t_0}^{t} d\,A\,x + f(t) - f(t_0)\,, \quad t \in [a,b]\,.$$

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Theorem (Schwabik, 1999)

ASSUME:

- $A \in \mathrm{BV}$ and $t_0 \in [a,b]$.
- $$\big[I - \Delta^- A(t)\big]^{-1} \in \mathcal{L}(X) \text{ for } t \in (t_0, b\,],$$
  $$\big[I + \Delta^+ A(s)\big]^{-1} \in \mathcal{L}(X) \text{ for } s \in [a, t_0).$$

THEN: For each $f \in \mathrm{G}$ and $\widetilde{x} \in X$, (L) has 1! solution $x \in \mathrm{G}$.

$$x_k(t) = \widetilde{x}_k + \int_a^t d[A_k]\,x + f_k(t) - f_k(a), \quad t \in [a,b].$$

$$x(t) = \widetilde{x} + \int_a^t d[A]\,x + f(t) - f(a), \quad t \in [a,b].$$

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$$A_k, A \in \mathrm{BV}, \quad f_k, f \in \mathrm{G}, \quad \widetilde{x}_k, \widetilde{x} \in X \quad \text{for } k \in \mathbb{N}\,.$$

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Theorem (Monteiro & M.T., 2013)

ASSUME:

- $\left[I - \Delta^- A(t)\right]^{-1} \in \mathcal{L}(X)$ for $t \in (a,b]$,
- $A_k \rightrightarrows A$ on $[a,b]$, $\alpha^* := \sup\{\mathrm{var}_a^b A_k : k \in \mathbb{N}\} < \infty$,
- $\widetilde{x}_k \to \widetilde{x}$, $f_k \rightrightarrows f$ on $[a,b]$.

THEN: $x_k \rightrightarrows x$ on $[a,b]$.

$$x_k' = P_k(t)\, x_k\,, \qquad x_k(a) = \widetilde{x}\,,$$
$$x' = \;P(t)\, x\,, \qquad x(a) = \widetilde{x}\,.$$

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Kurzweil & Vorel, 1957

ASSUME:

- $\|P_k\|_1 \le p^* < \infty$ for $k \in \mathbb{N}$,
- $\displaystyle\int_a^t P_k\, ds \rightrightarrows \int_a^t P\, ds.$

THEN: $x_k \rightrightarrows x$ on $[a, b]$.

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Opial, 1967

ASSUME:

- $\displaystyle\lim_{k\to\infty}\left[\left\|\int_a^t P_k\,ds - \int_a^t P\,ds\right\|_\infty \big(1+\|P_k\|_1\big)\right] = 0.$

THEN: $x_k \rightrightarrows x$ on $[a,b]$.

$$x_k(t) = \widetilde{x}_k + \int_a^t d[A_k]\, x_k(s) + f_k(t) - f_k(a), \quad t \in [a,b], \qquad \text{(L-k)}$$

$$x(t) = \widetilde{x} + \int_a^t d[A]\, x(s) + f(t) - f(a), \quad t \in [a,b]. \qquad \text{(L)}$$

Theorem (Monteiro & M.T., 2014)

ASSUME: $A_k \in \mathrm{BV}$, $f_k \in \mathrm{G}$, $\widetilde{x}_k \in X$ for $n \in \mathbb{N}$,

- $A \in \mathrm{BV}$, $f \in \mathrm{BV}$, $\widetilde{x} \in X$,
- $[I - \Delta^- A(t)]^{-1} \in L(X)$ for $t \in (a,b]$,
- $\lim_{k\to\infty} \big(1 + \mathrm{var}_a^b\, A_k\big)\|A_k - A\|_\infty = 0$,
- $\lim_{k\to\infty} \big(1 + \mathrm{var}_a^b\, A_k\big)\|f_k - f\|_\infty = 0$.

THEN: (L) has a unique solution $x \in \mathrm{BV}$ on $[a,b]$.

MOREOVER: (L-k) has a unique solution x_k for k sufficiently large and $x_k \rightrightarrows x$.

Let

$$A(t) = \begin{pmatrix} 0 & P(t) \\ Q(t) & 0 \end{pmatrix}, \quad f(t) = \begin{pmatrix} g(t) \\ h(t) \end{pmatrix}, \quad X = Y \times Y \quad \text{and} \quad \widetilde{x} = \begin{pmatrix} \widetilde{y} \\ \widetilde{z} \end{pmatrix},$$

where $P, Q \in \mathrm{BV}([a,b], L(Y))$ and $g, h \in \mathrm{BV}([a,b], Y)$, Y is a Banach space, $\widetilde{y}, \widetilde{z} \in Y$.

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$$x(t) = \widetilde{x} + \int_a^t d\,[A]\,x + f(t) - f(a)$$

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Then

$$x(t) = \widetilde{x} + \int_a^t d\,[A]\,x + f(t) - f(a)$$

reduces to

$$y(t) = \widetilde{y} + \int_a^t d[P]\,z + g(t) - g(a), \quad z(t) = \widetilde{z} + \int_a^t d[Q]\,y + h(t) - h(a)$$

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and $\quad [I_X - \Delta^- A(t)]^{-1} \in L(X) \quad$ iff

$$[I_Y - \Delta^- Q(t)\, \Delta^- P(t)]^{-1} \in L(Y) \quad (\text{or } [I_Y - \Delta^- P(t)\, \Delta^- Q(t)]^{-1} \in L(Y)) \quad \text{for } t \in (a,b]).$$

Consider systems

$$\left.\begin{aligned} y_k(t) &= \widetilde{y}_k + \int_a^t d[P_k]\, z_k + g_k(t) - g_k(a), \\ z_k(t) &= \widetilde{z}_k + \int_a^t d[Q_k]\, y_k + h_k(t) - h_k(a), \end{aligned}\right\} \tag{S-k}$$

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Corollary

ASSUME: $P, Q \in \mathrm{BV}([a,b], L(Y))$, $g, h \in \mathrm{BV}([a,b], Y)$, $\widetilde{y}, \widetilde{z} \in Y$,

- $[I_Y - \Delta^- Q(t)\, \Delta^- P(t)]^{-1} \in L(Y)$ or $[I_Y - \Delta^- P(t)\, \Delta^- Q(t)]^{-1} \in L(Y)$ for $t \in (a,b]$),
- $\lim_{k\to\infty} \|\widetilde{y}_k - \widetilde{y}\|_Y = 0, \qquad \lim_{k\to\infty} \|\widetilde{z}_k - \widetilde{z}\|_Y = 0,$
- $\lim_{k\to\infty} \big(1 + \mathrm{var}_a^b\, P_k + \mathrm{var}_a^b\, Q_k\big)\big(\|P_k - P\|_\infty + \|Q_k - Q\|_\infty\big) = 0,$
- $\lim_{k\to\infty} \big(1 + \mathrm{var}_a^b\, P_k + \mathrm{var}_a^b\, Q_k\big)\big(\|g_k - g\|_\infty + \|h_k - h\|_\infty\big) = 0.$

Consider systems

$$\left.\begin{aligned} y_k(t) &= \widetilde{y}_k + \int_a^t d[P_k]\, z_k + g_k(t) - g_k(a), \\ z_k(t) &= \widetilde{z}_k + \int_a^t d[Q_k]\, y_k + h_k(t) - h_k(a), \end{aligned}\right\} \tag{S-k}$$

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Corollary

ASSUME: $P, Q \in \mathrm{BV}([a,b], L(Y))$, $g, h \in \mathrm{BV}([a,b], Y)$, $\widetilde{y}, \widetilde{z} \in Y$,

- $[I_Y - \Delta^- Q(t)\, \Delta^- P(t)]^{-1} \in L(Y)$ or $[I_Y - \Delta^- P(t)\, \Delta^- Q(t)]^{-1} \in L(Y)$ for $t \in (a,b])$,
- $\lim_{k\to\infty} \|\widetilde{y}_k - \widetilde{y}\|_Y = 0$, $\lim_{k\to\infty} \|\widetilde{z}_k - \widetilde{z}\|_Y = 0$,
- $\lim_{k\to\infty} \big(1 + \mathrm{var}_a^b P_k + \mathrm{var}_a^b Q_k\big)\big(\|P_k - P\|_\infty + \|Q_k - Q\|_\infty\big) = 0$,
- $\lim_{k\to\infty} \big(1 + \mathrm{var}_a^b P_k + \mathrm{var}_a^b Q_k\big)\big(\|g_k - g\|_\infty + \|h_k - h\|_\infty\big) = 0$.

THEN:

- (S) has a unique solution $(y, z) \in BV([a,b], Y \times Y)$ on $[a,b]$,
- (S-k) has a unique solution $(y_k, z_k) \in G([a,b], Y \times Y)$ on $[a,b]$ for k sufficiently large,
- $\lim_{k\to\infty} \|y_k - y\|_\infty + \|z_k - z\|_\infty = 0$.

Meng and Zhang:

$$d\,y^{\bullet} + d\,[\mu_k(t)]\,y = 0, \quad y(0) = \widetilde{y},\ y^{\bullet}(0) = \widetilde{z},\ k \in \mathbb{N}\,, \qquad \text{(mz-k)}$$

where $\mu_k \in BV$ are right-continuous, $\widetilde{y}, \widetilde{z} \in \mathbb{R}$ and $y^{\bullet}$ is the generalized right-derivative of y.
They proved that the weak* convergence $\mu_k \to \mu$ yields

$$y_k \rightrightarrows y,\ y_k^{\bullet} \to y^{\bullet} \ \text{ in weak* topology and } \ y_k^{\bullet}(1) \to y^{\bullet}(1).$$

(S-k) reduce to (mz-k) when

$$[a,b] = [0,1],\ X = \mathbb{R},\ P_k(t) = t,\ Q_k(t) = \mu_k(t) \text{ and } \ g_k, h_k \text{ are constant.}$$

Similarly, (S) reduces to

$$d\,y^{\bullet} + d[\mu(t)]\,y = 0, \quad y(0) = \widetilde{y},\ y^{\bullet}(0) = \widetilde{z} \qquad \text{(mz)}$$

if

$$P(t) = t,\ Q(t) = \mu(t) \ \text{ and } \ g, h \ \text{ are constant.}$$

As existence conditions are obviously satisfied, by our **Corollary** we have

$$\lim_{k\to\infty} (\|y_k - y\|_\infty + \|y_k^{\bullet} - y^{\bullet}\|_\infty) = 0.$$

whenever

$$\lim_{k\to\infty} (1 + \operatorname{var}_0^1 \mu_k)\|\mu_k - \mu\|_\infty = 0.$$

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$\sigma(t) := \inf \left((t, b] \cap \mathbb{T}\right)$ is the **forward jump operator**,

$\rho(t) := \sup \left([a, t) \cap \mathbb{T}\right)$ is the **backward jump operator**

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$\mu(t) = \sigma(t) - t$ is the **graininess** of the time scale.

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For a given $\delta > 0$, a division $D = \{\alpha_0, \alpha_1, \dots, \alpha_{\nu(D)}\} \subset [a,b]_{\mathbb{T}}$ of $[a,b]$ is said to be δ**-fine** if

$$\text{either} \quad \alpha_i - \alpha_{i-1} < \delta \quad \text{or} \quad \rho(\alpha_i) = \alpha_{i-1}.$$

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We also say that $P = (D, \xi)$ is a **tagged division** of $[a,b]_{\mathbb{T}}$ if

$$\xi = \{\xi_1, \ldots \xi_{\nu(D)}\} \quad \text{and} \quad \xi_i \in [\alpha_{i-1}, \alpha_i) \cap \mathbb{T} \quad \text{for } i \in \{1, \ldots, \nu(D)\}.$$

Then

$$I = \int_a^b f(t)\,\Delta t$$

iff for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\Big|\sum_{i=1}^{\nu(D)} f(\xi_i)(\alpha_i - \alpha_{i-1}) - I\Big| < \varepsilon \quad \text{for all } \delta\text{−fine tagged divisions } P = (D, \xi) \text{ of } [a,b]_{\mathbb{T}}.$$

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Linear dynamical equations on time scales

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Proposition (Slavík)

ASSUME: $f : [a,b]_{\mathbb{T}} \to \mathbb{R}^n$ is rd-continuous,

$$F_1(t) = \int_a^t f(s)\,\Delta s \quad \text{and} \quad F_2(t) = \int_a^t f(\widetilde{\sigma}(s))\; d[\widetilde{\sigma}(s)] \quad \text{for } t \in [a,b].$$

THEN: $F_2 = F_1 \circ \widetilde{\sigma}$.

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THEN: $F_2 = F_1 \circ \widetilde{\sigma}$.

Consider equation

$$y(t) = \widetilde{y} + \int_a^t \big[P(s)\,y(s) + h(s)\big]\,\Delta s\,, \quad t \in [a,b]_{\mathbb{T}}\,, \tag{D}$$

where $P : [a,b]_{\mathbb{T}} \to \mathcal{L}(\mathbb{R}^n)$ and $h : [a,b]_{\mathbb{T}} \to \mathbb{R}^n$ are rd-continuous on $[a,b]_{\mathbb{T}}$, and put

Put $\widetilde{\sigma}(t) := \inf\big([t,b]\cap\mathbb{T}\big)$ (recall: $\sigma(t) := \inf\big((t,b]\cap\mathbb{T}\big)$).

Proposition (Slavík)

ASSUME: $f:[a,b]_{\mathbb{T}} \to \mathbb{R}^n$ is rd-continuous,

$$F_1(t) = \int_a^t f(s)\,\Delta s \quad\text{and}\quad F_2(t) = \int_a^t f(\widetilde{\sigma}(s))\, d[\widetilde{\sigma}(s)] \quad \text{for } t\in[a,b].$$

THEN: $F_2 = F_1 \circ \widetilde{\sigma}$.

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where $P:[a,b]_{\mathbb{T}}\to\mathcal{L}(\mathbb{R}^n)$ and $h:[a,b]_{\mathbb{T}}\to\mathbb{R}^n$ are rd-continuous on $[a,b]_{\mathbb{T}}$, and put

$$A(t) = \int_a^t P(\widetilde{\sigma}(s))\,d[\widetilde{\sigma}(s)] \quad \text{a} \quad f(t) = \int_a^t h(\widetilde{\sigma}(s))\,d[\widetilde{\sigma}(s)] \quad \text{for } t\in[a,b]\,.$$

Put $\widetilde{\sigma}(t) := \inf\big([t,b]\cap\mathbb{T}\big)$ (recall: $\sigma(t) := \inf\big((t,b]\cap\mathbb{T}\big)$).

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<u>Then</u>: $F_2 = F_1\circ\widetilde{\sigma}$.

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Theorem (Slavík)

- If $y:[a,b]_{\mathbb{T}}\to\mathbb{R}^n$ is a solution of (LD), then $x=y\circ\widetilde{\sigma}$ is a solution of

$$x(t)=\widetilde{y}+\int_a^t d[A]\,x+f(t)-f(a)\,, \quad t\in[a,b]\,. \tag{L}$$

- If x is a solution of (GL) and $y=x|_{\mathbb{T}}$, then y is a solution of (LD).

$$y(t) = \tilde{y} + \int_a^t \big[P(s)\,y(s) + h(s)\big]\,\Delta s\,, \quad t \in [a,b]_{\mathbb{T}}\,, \tag{LD}$$

$$y(t) = \tilde{y}_k + \int_a^t \big[P_k(s)\,y(s) + h_k(s)\big]\,\Delta s\,, \quad t \in [a,b]_{\mathbb{T}}\,, \tag{LD-k}$$

Corollary

ASSUME: P, $P_k\colon [a,b]_{\mathbb{T}} \to \mathcal{L}(\mathbb{R}^n)$, h, $h_k\colon [a,b]_{\mathbb{T}} \to \mathbb{R}^n$ for $k \in \mathbb{N}$ are rd-continuous in $[a,b]_{\mathbb{T}}$,

$$\alpha_k = \sup_{t\in[a,b]_{\mathbb{T}}} \|P_k(t)\|_{L(\mathbb{R}^n)} + \sup_{t\in[a,b]_{\mathbb{T}}} \|h_k(t)\|_{\mathbb{R}^n} \quad \text{for } k \in \mathbb{N}\,,$$

$$\lim_{k\to\infty} \|\tilde{y}_k - \tilde{y}\|_{\mathbb{R}^n} = 0\,,$$

$$\lim_{k\to\infty} \sup_{t\in[a,b]_{\mathbb{T}}} \Big\| \int_a^t (P_k(s) - P(s))\,\Delta s \Big\|_{L(\mathbb{R}^n)} \big[1 + \alpha_k\big] = 0\,,$$

$$\lim_{k\to\infty} \sup_{t\in[a,b]_{\mathbb{T}}} \Big\| \int_a^t (h_k(s) - h(s))\,\Delta s \Big\|_{L(\mathbb{R}^n)} \big[1 + \alpha_k\big] = 0\,.$$

THEN: (LD) has a solution y, (LD-k) has a solution y_k for $k \in \mathbb{N}$ sufficiently large and

$$\lim_{k\to\infty} \sup_{t\in[a,b]_{\mathbb{T}}} \|y_k(t) - y(t)\|_{\mathbb{R}^n} = 0\,.$$

A sequence $\{f_n, g_n\}$ is **equi-integrable** if:

- $\displaystyle\int_a^b f_n \, dg_n$ exists for each $n \in \mathbb{N}$,
- for every $\varepsilon > 0$, there is a gauge δ on $[a, b]$ such that

$$\left| \int_a^b f_n \, dg_n - S(f_n, dg_n, P) \right| < \varepsilon$$

holds for each δ-fine partition P of $[a, b]$ and for every $n \in \mathbb{N}$.

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Equi-integrability Convergence Theorem

ASSUME: $\{f_n, g_n\}$ is equi-integrable,

$$f_n(t) \to f(t) \quad \text{and} \quad g_n(t) \to g(t) \quad \text{for } t \in [a, b].$$

THEN:
$$\int_a^b f \, dg = \lim_{n \to \infty} \int_a^b f_n \, d g_n.$$

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MOREOVER: $\int_a^t f_n \, d g_n \rightrightarrows \int_a^t f \, d g$ on $[a, b]$ whenever $\{g_n\}$ is uniformly bounded on $[a, b]$.

Preiss-Schwabik-Kurzweil (PSK) Convergence Theorem

ASSUME: $\dim X < \infty$, $g \in \mathrm{BV}$ and f, $\{f_n\}$ are such that

(i) $\displaystyle\int_a^b f_n\, dg$ exists for every $n \in \mathbb{N}$,

(ii) $\displaystyle\lim_{n\to\infty} f_n(t) = f(t)$ for every $t \in [a,b]$,

(iii) $\displaystyle\Big|\sum_{j=1}^{\ell} \int_{\sigma_{j-1}}^{\sigma_j} f_{m_j}\, dg\Big| \le K < \infty$ for $\ell \in \mathbb{N}$, $\{\sigma_0, \dots, \sigma_\ell\} \in \mathcal{D}$ and $\{m_i\}_{i=1}^{\ell} \subset \mathbb{N}^\ell$.

THEN: $\{f_n, g\}$ is **equi-integrable** and $\displaystyle\int_a^t f_n\, dg \rightrightarrows \int_a^t f\, dg$.

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THEN: $\{f_n, g\}$ is **equi-integrable** and $\displaystyle\int_a^t f_n\, dg \rightrightarrows \int_a^t f\, dg$.

If $\dim X < \infty$, then Bounded Convergence Theorem follows from PSK Theorem.

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If $\dim X < \infty$, then Bounded Convergence Theorem follows from PSK Theorem.

Open questions:

- $\dim X = \infty$,
- $\{g_n\}$ instead of g.

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Kurzweil-Stieltjes integral and its applications.

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