Quasilinear elliptic equations with gradient dependence

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[1]joint work with D. Averna and D. Motreanu

Nonlinear Dirichlet problem driven the (p, q)-Laplacian operator

$$\begin{cases} -\Delta_p u - \mu \Delta_q u = f(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \qquad (P_\mu)$$

- $\Omega \subset \mathbb{R}^N$ is a nonempty bounded open set with the boundary $\partial\Omega$
- μ positive real parameter
- $1 < q < p$,
- $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2} \nabla u)$
  $\Delta_q u = \mathrm{div}(|\nabla u|^{q-2} \nabla u)$,
- $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, is a Carathéodory function
  - $f(\cdot, s, \xi)$ is measurable for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$
  - $f(x, \cdot, \cdot)$ is continuous for a.e. $x \in \Omega$.

The case in which the nonlinear term does not depend on the gradient ∇u

$$\begin{cases} -\Delta_p u - \mu \Delta_q u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has been studied by using variational methods

- S. Marano - S. Mosconi- N. Papageorgiou, *Multiple Solutions to (p,q)-Laplacian Problems with Resonant Concave Nonlinearity*, Adv. Nonlinear Stud. (2016), **16** (1) 51–65.

- D. Mugnai - N. Papageorgiou, *Wang's Multiplicity result for superlinea (p,q)-equations without the Ambrosetti-Rabinowitz Condition*, Transactions of the American Mathematical Soc. (2015) **366** (9) 4919–4936.

If $\mu = 0$

$$\begin{cases} -\Delta_p u = f(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad (P_0)$$

F. Faraci - D Motreanu - D. Puglisi, *Positive solutions of quasi-linear elliptic equations with dependence on the gradient*, Calc. Var. **54** (2015) 525–538.

when $\mu = 1$

$$\begin{cases} -\Delta_p u - \Delta_q u = f(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

L.F.O. Faria - O.H. Miyagaki - D. Motreanu - M. Tanaka, *Existence results for nonlinear elliptic equations with Leray-Lions operator and dependence on the gradient*, **96** (2014) 154–166.

Under suitable assumption on f we want to prove

- Existence of solutions by using the theory of pseudomonotone operator
- Asymptotic properties as $\mu \to 0^+$ and $\mu \to +\infty$
- Uniqueness of solutions
- Location of solutions by using the method of sub-solution and super-solution for quasilinear elliptic equations combined with comparison arguments.

We refer to following books for details related to pseudomonotone operator and to the method of subsolution-supersolution.

S. Carl - V. K. Le - D. Motreanu, Nonsmooth Variational Problems and Their Inequalities Comparison Principles and Applications, *Springer Monographs in Mathematics*, Springer, New York (2007).

D. Motreanu - V. V. Motreanu - N. Papageorgiou, Topological and Variational Methods with Applications to Nonlinear Boundary Value Problems, *Springer, New York* (2014).

Definition

Let $A : X \to X^*$. We say that A has S_+-property iff every sequence $\{u_n\} \subset X$ such that $u_n \rightharpoonup u$ in X and $\limsup_{n\to+\infty}\langle Au_n, u_n - u\rangle \le 0$ implies that $u_n \to u$ in X.

Definition

$A : X \to X^*$ is called
pseudomonotone if $u_n \rightharpoonup u$ and $\limsup_{n\to+\infty}\langle Au_n, u_n - u\rangle \le 0$ imply that $Au_n \rightharpoonup Au$ and $\langle Au_n, u_n\rangle \to \langle Au, u\rangle$.

Consider the negative p-Laplacian

$$-\Delta_p : W_0^{1,p}(\Omega) \to W_0^{-1,p'}(\Omega)$$

is continuous, bounded, pseudomonotone and has the S_+-property

the first eigenvalue of p-Laplacian operator admits the following variational characterization

$$\lambda_{1p} = \inf_{u \in W_0^{1,p}(\Omega)\ u \neq 0} \frac{\|\nabla u\|^p_{L^p(\Omega)}}{\|u\|^p_{L^p(\Omega)}}$$

The nonlinearity f safisfies the following conditions

(H1) There exist constants $a_1 \geq 0$, $a_2 \geq 0$, $\alpha \in [0, p^* - 1[$, $\beta \in [0, \frac{p}{(p^*)'}[$ and a function $\sigma \in L^{\gamma'}(\Omega)$, with $\gamma \in [1, p^*[$, such that

$$|f(x,s,\xi)| \leq \sigma(x) + a_1|s|^\alpha + a_2|\xi|^\beta \text{ a.e. } x \in \Omega,\ \forall (s,\xi) \in \mathbb{R} \times \mathbb{R}^N;$$

(H2) there exist constants $d_1 \geq 0$, $d_2 \geq 0$ with $\lambda_{1,p}^{-1} d_1 + d_2 < 1$, and a function $\omega \in L^1(\Omega)$ such that

$$f(x,s,\xi)s \leq \omega(x) + d_1|s|^p + d_2|\xi|^p \text{ a.e. } x \in \Omega,\ \forall (s,\xi) \in \mathbb{R} \times \mathbb{R}^N.$$

$$p' = \frac{p}{p-1} \qquad p^* = \begin{cases} \frac{pN}{N-p} & p < N \\ +\infty & p \geq N \end{cases}$$

The functional space associated to problem is the Sobolev space $W_0^{1,p}(\Omega)$ with the norm

$$||u|| := \left(\int_\Omega |\nabla u|^p dx \right)^{\frac{1}{p}}$$

for all $u \in W_0^{1,p}(\Omega)$.

A (weak) solution of problem (P_μ) for $\mu \geq 0$ is any $u \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v \, dx + \mu \int_\Omega |\nabla u|^{q-2} \nabla u \nabla v \, dx - \int_\Omega f(x, u, \nabla u) v \, dx = 0$$

for all $v \in W_0^{1,p}(\Omega)$

The Nemytskii operator associated to f

$$N : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$$

defined by

$$N(u) = f(x, u, \nabla u)$$

is well defined, continuous and bounded.

We consider the operator

$$A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$$

$$A(u) = -\Delta_p u - \mu \Delta_q u - N(u), \tag{1}$$

Then
$u \in W_0^{1,p}(\Omega)$ is a weak solution of problem $(P_\mu) \Longleftrightarrow A(u) = 0$

Theorem

Main theorem on pseudomonotone operator *Let X be a real reflexive Banach space, $A : X \to X^*$ be a pseudomonotone, bounded and coercive operator. Then there is a solution of the equation $Ax = 0$.*

Existence of solutions

Theorem

Assume that conditions (H1) and (H2) hold. Then problem (P_μ), *with* $\mu \geq 0$, *admits at least one weak solution* $u_\mu \in W_0^{1,p}(\Omega)$.

Proof.

. $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$

$$A(u) = -\Delta_p u - \mu\Delta_q u - N(u),$$

1. $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is bounded, which means that it maps bounded sets onto bounded sets.
2. for every sequence $\{u_n\} \subset W_0^{1,p}(\Omega)$ such that $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, by using that the operator $-\Delta_p - \mu\Delta_q$ on the space $W_0^{1,p}(\Omega)$ has the S_+-property, we have $u_n \to u$.
3. A is pseudomonotone
4. A is coercive

$$\lim_{\|u\|\to\infty} \frac{\langle Au, u\rangle}{\|u\|} = +\infty.$$

Since $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is pseudomonotone, bounded and coercive, we can apply the main theorem on pseudomonotone operators . Therefore there is at least one element $u_\mu \in W_0^{1,p}(\Omega)$ such that $Au_\mu = 0$, so u_μ is a weak solution of problem (P_μ), which completes the proof. □

Problem (P_μ) possesses a solution $u_\mu \in W_0^{1,p}(\Omega)$ for every $\mu > 0$. We establish the following a priori estimate.

Lemma

Assume that conditions (H1) and (H2) hold. Then there exists a constant $b > 0$ independent of $\mu > 0$ such that

$$\|\nabla u_\mu\|_{L^p(\Omega)} \leq b, \ \forall \mu > 0. \tag{2}$$

where

$$b = \left(\frac{\|\omega\|_{L^1(\Omega)}}{1 - d_1 \lambda_{1p}^{-1} - d_2} \right)^{\frac{1}{p}}$$

Asymptotic properties as $\mu \to 0$

We consider the limit points (u_μ) as $\mu \to 0$ in problem (P_μ).

Theorem

For any sequence $\mu_n \to 0^+$, there exists a relabeled subsequence of solutions (u_{μ_n}) of the corresponding problems (P_{μ_n}) such that $u_{\mu_n} \to u$ in $W_0^{1,p}(\Omega)$, with $u \in W_0^{1,p}(\Omega)$ weak solution of problem (P_0)

$$\begin{cases} -\Delta_p u = f(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Proof.
Since u_n is a weak solution of problem (P_{μ_n}), from Lemma
$\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Then there exists a subsequence $\{u_{k_n}\}$ such that $u_{k_n} \rightharpoonup u$. From $(\mathrm{P}_{\mu_{k_n}})$ we obtain

$$\lim_{k_n \to +\infty} \langle -\Delta_p u_{k_n}, u_{k_n} - u \rangle = 0$$

Since $-\Delta_p$ satisfies the S_+-property, we have $u_{k_n} \to u$.
Letting $k_n \to +\infty$ in $(\mathrm{P}_{\mu_{k_n}})$ it is easy to see that u is a weak solution of problem (P_0)

Asymptotic properties as $\mu \to +\infty$

We consider $\mu \to +\infty$ and the problem

$$\begin{cases} -\frac{1}{\mu}\Delta_p u - \Delta_q u = \frac{1}{\mu} f(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad (P_{\frac{1}{\mu}})$$

observe that the solutions of problem (P_μ) are solutions of problem $(P_{\frac{1}{\mu}})$.

Theorem

For any sequence $\mu_n \to +\infty$, the sequence of solutions (u_{μ_n}) of the corresponding problems (P_{μ_n}) satisfies $u_{\mu_n} \to 0$ in $W_0^{1,q}(\Omega)$.

Proof.
Proceeding as in the proof of previous Theorem, we set $u_n := u_{\mu_n}$ and apply Lemma to derive that the sequence (u_n) is bounded in $W_0^{1,p}(\Omega)$, so up to a relabeled subsequence we have $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ for some $u \in W_0^{1,p}(\Omega)$.
We note that u_n satisfies

$$\begin{cases} -\frac{1}{\mu_n}\Delta_p u_n - \Delta_q u_n = \frac{1}{\mu_n} f(x, u_n, \nabla u_n) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

If we act with $u_n - u$ in (3), we find that

$$\lim_{n\to+\infty} \langle -\Delta_q u_n, u_n - u \rangle = 0.$$

The S_+-property of the operator $-\Delta_q : W_0^{1,q}(\Omega) \to W^{-1,q'}(\Omega)$ guarantees that $u_n \to u$ in $W_0^{1,q}(\Omega)$. Letting $n \to \infty$ in (3) entails $\Delta_q u = 0$, so $u = 0$.

Uniqueness result

We illustrate this topic by presenting a uniqueness result in the case where $p = 2$ or $q = 2$. Our assumption is as follows:

(U1) there exists a constant $b_1 \geq 0$ such that

$$(f(x,s,\xi)-f(x,t,\xi))(s-t) \leq b_1|s-t|^2 \text{ a.e. } x \in \Omega,\ \forall \xi \in \mathbb{R}^N,\ \forall s,t \in \mathbb{R};$$

(U2) there exist a function $\tau \in L^{\delta}(\Omega)$, with some $\delta \in [1, p^*[$, and a constant $b_2 \geq 0$ such that the function $f(x,s,\cdot) - \tau(x)$ is linear and

$$|f(x,s,\xi) - \tau(x)| \leq b_2|\xi| \text{ a.e. } x \in \Omega,\ \forall (s,\xi) \in \mathbb{R} \times \mathbb{R}^N.$$

Theorem

Assume that conditions (H1), (H2), (U1) and (U2) hold.

(i) *If* $p = 2 > q > 1$ *and*

$$b_1\lambda_{1,2}^{-1} + b_2\lambda_{1,2}^{-\frac{1}{2}} < 1$$

then the solution of problem (P_μ) *is unique for every* $\mu > 0$.

(ii) *If* $p > q = 2$, *then the solution of problem* (P_μ) *is unique for every*

$$\mu > b_1\lambda_{1,2}^{-1} + b_2\lambda_{1,2}^{-\frac{1}{2}}$$

.

Proof.

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Suppose that $v_\mu \in W_0^{1,p}(\Omega)$ is a second solution of (P_μ). Acting with $u_\mu - v_\mu$ on the equation in (P_μ) gives

$$\begin{aligned} &\langle -\Delta_p u_\mu + \Delta_p v_\mu, u_\mu - v_\mu\rangle + \mu\langle -\Delta_q u_\mu + \Delta_q v_\mu, u_\mu - v_\mu\rangle \\ &= \int_\Omega (f(x, u_\mu, \nabla u_\mu) - f(x, v_\mu, \nabla u_\mu)(u_\mu - v_\mu)\ dx \\ &+ \int_\Omega (f(x, v_\mu, \nabla u_\mu) - f(x, v_\mu, \nabla v_\mu)(u_\mu - v_\mu)\ dx. \end{aligned} \tag{4}$$

(i) For $p = 2$, hypotheses (U1) and (U2), in conjunction with (4) and the monotonicity of $-\Delta_q$, imply

$$\begin{aligned} &\|\nabla(u_\mu - v_\mu)\|^2_{L^2(\Omega)} \le b_1\|u_\mu - v_\mu\|^2_{L^2(\Omega)} + \int_\Omega (f(x, v_\mu, \nabla(\tfrac{1}{2}(u_\mu - v_\mu)^2)\ dx \\ &\le (b_1\lambda_{1,2}^{-1} + \tfrac{b_2}{2})\|\nabla(u_\mu - v_\mu)\|^2_{L^2(\Omega)}. \end{aligned}$$

Using that $b_1\lambda_{1,2}^{-1} + b_2\lambda_{1,2}^{-\frac{1}{2}} < 1$, the equality $u_\mu = v_\mu$ follows.
(ii) For $p > q = 2$, arguing as in the case of part (i), we find the estimate

$$\mu\|\nabla(u_\mu - v_\mu)\|^2_{L^2(\Omega)} \le (b_1\lambda_{1,2}^{-1} + b_2\lambda_{1,2}^{-\frac{1}{2}})\|\nabla(u_\mu - v_\mu)\|^2_{L^2(\Omega)}.$$

The conclusion that $u_\mu = v_\mu$ ensues provided that $b_1\lambda_{1,2}^{-1} + \frac{b_2}{2} < \mu$. □

Location of solutions

Our main goal is to obtain a solution $u \in W_0^{1,p}(\Omega)$ of problem (P_μ) with the location property $\underline{u} \leq u \leq \overline{u}$ a.e. in Ω, where $\underline{u}$ and $\overline{u}$ are subsolution and supersolution of problem (P_μ).

$\overline{u} \in W^{1,p}(\Omega)$ is a *supersolution* for problem (P_μ) if $\overline{u} \geq 0$ on $\partial\Omega$ and

$$\int_\Omega \left(|\nabla \overline{u}|^{p-2} \nabla \overline{u} + \mu |\nabla \overline{u}|^{q-2} \nabla \overline{u}\right) \nabla v \, dx \geq \int_\Omega f(x, \overline{u}, \nabla \overline{u}) v \, dx$$

for all $v \in W_0^{1,p}(\Omega)$, $v \geq 0$ a.e. in Ω.

$\underline{u} \in W^{1,p}(\Omega)$ is a *subsolution* for problem (P_μ) if $\underline{u} \leq 0$ on $\partial\Omega$ and

$$\int_\Omega \left(|\nabla \underline{u}|^{p-2} \nabla \underline{u} + \mu |\nabla \underline{u}|^{q-2} \nabla \underline{u}\right) \nabla v \, dx \leq \int_\Omega f(x, \underline{u}, \nabla \underline{u}) v \, dx$$

for all $v \in W_0^{1,p}(\Omega)$, $v \geq 0$ a.e. in Ω.

Given a subsolution $\underline{u} \in W^{1,p}(\Omega)$ and a supersolution $\overline{u} \in W^{1,p}(\Omega)$ for problem (P_μ) with $\underline{u} \leq \overline{u}$ a.e. in Ω, we assume that $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies the growth condition:

(H) There exist a function $\sigma \in L^{\gamma'}(\Omega)$ for $\gamma' = \frac{\gamma}{\gamma-1}$ with $\gamma \in (1, p^*)$ and constants $a > 0$ and $\beta \in [0, \frac{p}{(p^*)'})$ such that

$$|f(x, s, \xi)| \leq \sigma(x) + a|\xi|^\beta \text{ for a.e. } x \in \Omega, \text{ all } s \in [\underline{u}(x), \overline{u}(x)], \ \xi \in \mathbb{R}^N.$$

Theorem

Let $\underline{u}$ and $\overline{u}$ be a subsolution and a supersolution of problem (P_μ), respectively, with $\underline{u} \leq \overline{u}$ a.e. in Ω such that hypothesis (H) is fulfilled. Then problem (P_μ) possesses a solution $u \in W_0^{1,p}(\Omega)$ satisfying the location property $\underline{u} \leq u \leq \overline{u}$ a.e. in Ω.

Proof.

- Consider auxiliary truncated problem depending on a positive parameter λ (for any fixed $\mu \geq 0$)

$$(T_{\lambda,\mu}) \qquad -\Delta_p u - \mu\Delta_q u + \lambda B(u) = N(Tu).$$

where T is the truncation operator $T : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ defined by

$$Tu(x) = \begin{cases} \overline{u}(x) & \text{if} \quad u(x) > \overline{u}(x) \\ u(x) & \text{if} \quad \underline{u}(x) \leq u(x) \leq \overline{u}(x) \\ \underline{u}(x) & \text{if} \quad u(x) < \underline{u}(x), \end{cases}$$

which is known to be continuous and bounded.
π is the cut-off function $\pi : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$\pi(x,s) = \begin{cases} (s - \overline{u}(x))^{\frac{\beta}{p-\beta}} & \text{if} \quad s > \overline{u}(x) \\ 0 & \text{if} \quad \underline{u}(x) \leq s \leq \overline{u}(x) \\ -(\underline{u}(x) - s)^{\frac{\beta}{p-\beta}} & \text{if} \quad s < \underline{u}(x). \end{cases}$$

$B : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is the Nemytskij operator given by $B(u) = \pi(\cdot, u(\cdot))$

- $N : [\underline{u}, \overline{u}] \to W^{-1,p'}(\Omega)$ is the Nemytskij operator determined by the function f in (P_μ), that is

$$N(u)(x) = f(x, u(x), \nabla u(x)),$$

- for $\lambda > 0$ sufficiently large, there is a solution $u \in W_0^{1,p}(\Omega)$ of problem $(T_{\mu,\lambda})$.
- by using comparison arguments we prove that every solution $u \in W_0^{1,p}(\Omega)$ of problem $(T_{\mu,\lambda})$ $\underline{u} \le u \le \overline{u}$ a.e. in Ω.
- the solution u of the auxiliary truncated problem $(T_{\lambda,\mu})$ satisfies $Tu = u$ and $B(u) = 0$, so it is a solution of the original problem (P_μ)

Positive solutions

We want to show you a result on the existence of positive solutions to problem (P_μ).

The idea is to construct a subsolution $\underline{u} \in W^{1,p}(\Omega)$ and a supersolution $\overline{u} \in W^{1,p}(\Omega)$ with $0 < \underline{u} \leq \overline{u}$ a.e. in Ω for which previous Theorem can be applied.

we suppose the following assumptions on f

($H3$) There exist constants $a_0 > 0$, $b > 0$, $\delta > 0$ and $r > 0$, with $r < p - 1$ if $\mu = 0$ and $r < q - 1$ if $\mu > 0$, such that

$$\left(\frac{a_0}{b}\right)^{\frac{1}{p-r-1}} < \delta \tag{5}$$

and

$$f(x, s, \xi) \geq a_0 s^r - b s^{p-1} \text{ for a.e. } x \in \Omega, \text{ all } 0 < s < \delta, \ \xi \in \mathbb{R}^N. \tag{6}$$

($H4$) There exists a constant $s_0 > \delta$, with $\delta > 0$ in ($H3$), such that

$$f(x, s_0, 0) \leq 0 \quad \text{for a.e. } x \in \Omega. \tag{7}$$

Our result on the existence of positive solutions for problem (P_μ) is as follows.

Theorem

Assume $(H3)$, $(H4)$ *and that*

$$|f(x,s,\xi)| \leq \sigma(x) + a|\xi|^{\beta} \quad \text{for a.e. } x \in \Omega, \ \text{all } s \in [0, s_0], \ \xi \in \mathbb{R}^N,$$

with a function $\sigma \in L^{\gamma'}(\Omega)$ *for* $\gamma \in [1, p^*)$ *and constants* $a > 0$, $\beta \in [0, \frac{p}{(p^*)'})$, *and* s_0 *in* $(H4)$. *Then, for every* $\mu \geq 0$, *problem* (P_μ) *possesses a positive smooth solution* u *satisfying the a priori estimate* $u(x) \leq s_0$ *for all* $x \in \Omega$.

Proof.

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1. Consider the following auxiliary problem

$$\begin{cases} -\Delta_p u - \mu\Delta_q u + b|u|^{p-2}u = a_0(u^+)^r & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{8}$$

2. We prove that there exists a solution $\underline{u} \in C_0^1(\overline{\Omega})$ of problem such that $\underline{u} > 0$ in Ω.
3. We claim that $\underline{u}$ is a subsolution for problem (P_μ).
4. Hypothesis $(H4)$ guarantees that $\overline{u} = s_0$ is a supersolution of problem (P_μ).
5. We have $\underline{u} < \overline{u}$ in Ω.
6. The hypothesis (H) is verified by constructed pair $(\underline{u}, \overline{u})$ of subsolution-supersolution for problem (P_μ). Therefore previous theorem ensuring the existence of a solution $u \in W_0^{1,p}(\Omega)$ for the problem (P_μ), which satisfies the enclosure property $\underline{u} \leq u \leq \overline{u}$ a.e. in Ω.
7. Taking into account that $\underline{u} > 0$, we conclude that the solution u is positive.

References

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