Modified midpoint method for delay differential equations - stability analysis

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We consider

$$y'(t) = ay(t) + by(t-\tau), \qquad t \geq 0, \qquad \text{(DDE)}$$

where $a, b, \tau \in \mathbb{R}$, $\tau > 0$.

Initial condition

$$y(t) = \phi(t), \qquad t \in [-\tau, 0]\,. \qquad \text{(IC)}$$

Notion of asymptotic stability of DDE

$$y'(t) = ay(t) + by(t - \tau), \qquad t \geq 0, \qquad \text{(DDE)}$$

Delay differential equation (DDE) is said to be asymptotically stable (**AS**) if all its solutions satisfy

$$\lim_{t\to\infty} y(t) = 0\,.$$

$$y'(t) = ay(t) + by(t-\tau), \qquad t \geq 0, \qquad \text{(DDE)}$$

Theorem

Equation (DDE) *is asymptotically stable if and only if*

$$a \leq b < -a \qquad \textit{for all} \qquad \tau > 0 \qquad (1)$$

and in addition to the previous

$$|a| + b < 0 \qquad \textit{for} \qquad \tau < \frac{\arccos(-a/b)}{(b^2 - a^2)^{1/2}}.$$

Andronov A.A., Mayer A.G.: The simplest linear systems with delay. *Autom. Remote Control*, **7(2–3)**, (1946), 95–106.

Hayes N.D.: **Roots of the transcendental equations associated with certain difference-differential equation**,

J. London. Math. Soc. **25**, (1950), 226–232.

$$y'(t) = ay(t) + by(t-\tau), \qquad t \geq 0, \qquad \text{(DDE)}$$

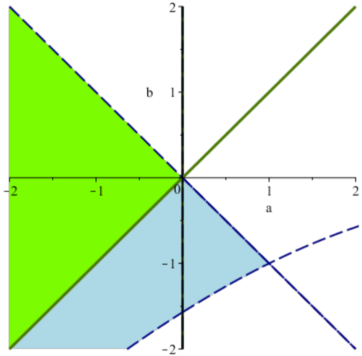


Figure: $a \leq b < -a$ for all $\tau > 0$; $|a| + b < 0$ for $\tau < \frac{\arccos(-a/b)}{(b^2-a^2)^{1/2}}$; $[\tau = 1]$

Discrete case

$$Y(n+2) + \alpha Y(n) + \beta Y(n-\ell) = 0, \quad n = 0, 1, 2, \dots, \qquad \text{(DC)}$$

where $\alpha,\ \beta \in \mathbb{R}$ and $\ell \in \mathbb{N}$.

Difference equation (DC) is said to be asymptotically stable (**AS**) if all its solutions satisfy

$$\lim_{n\to\infty} Y(n) = 0\,.$$

Preliminary result - three-term difference equation

$$Y(n+2) + \alpha Y(n) + \beta Y(n-\ell) = 0, \quad n = 0, 1, 2, \ldots . \qquad \text{(DC)}$$

Theorem

Let α, β be arbitrary reals such that $\alpha\beta \neq 0$.

(i) *Let ℓ be even and $\beta(-\alpha)^{\ell/2+1} < 0$. Then (DC) is asymptotically stable if and only if*

$$|\alpha| + |\beta| < 1 . \tag{2}$$

(ii) *Let ℓ be even and $\beta(-\alpha)^{\ell/2+1} > 0$. Then (DC) is asymptotically stable if and only if either*

$$|\alpha| + |\beta| \leq 1 , \tag{3}$$

or

$$||\alpha| - |\beta|| < 1 < |\alpha| + |\beta|, \quad \ell < 2 \arccos \frac{\alpha^2 + \beta^2 - 1}{2|\alpha\beta|} \Big/ \arccos \frac{\alpha^2 - \beta^2 + 1}{2|\alpha|} \tag{4}$$

holds.

Čermák J, Tomášek P: On delay-dependent stability conditions for a three-term linear difference equation. *Funkcial. Ekvac.* **57(1)**, (2014), 91–106.

Preliminary result - three-term difference equation

$$Y(n+2)+\alpha Y(n)+\beta Y(n-\ell)=0, \quad n=0,1,2,\dots, \text{(DC)}$$

where α, $\beta \in \mathbb{R}$ and $\ell \in \mathbb{N}$.

Theorem

(i) *Let ℓ be odd and $\alpha < 0$. Then (DC) is asymptotically stable if and only if*

$$|\alpha|+|\beta|<1\,.$$

holds.

(ii) *Let ℓ be odd and $\alpha > 0$. Then (DC) is asymptotically stable if and only if either*

$$|\alpha|+|\beta|\leq 1\,,$$

or

$$\beta^2<1-\alpha<|\beta|, \quad \ell<2\arcsin\frac{1-\alpha^2-\beta^2}{2|\alpha\beta|}\Big/\arccos\frac{\alpha^2-\beta^2+1}{2|\alpha|} \tag{5}$$

holds.

Čermák J, Tomášek P: On delay-dependent stability conditions for a three-term linear difference equation.

Funkcial. Ekvac. **57(1)**, (2014), 91–106.

Numerical scheme

We consider

$$y'(t) = ay(t) + by(t-\tau), \qquad t > 0,$$

and

$$Y(n+2) - \frac{1+ah}{1-ah}Y(n) - \frac{2bh}{1-ah}Y(n-k+1) = 0, \quad n = 0,1,\ldots. \quad \text{(NS)}$$

- equidistant mesh: $t_n = nh$, $n = 0, 1\ldots$
- stepsize $h = \tau/k$, where $k \geq 2$, $k \in \mathbb{Z}$
- $ah \neq 1$.
- $Y(n) \approx y(t_n)$, $n = 0, 1, 2, \ldots$

Remark

Such efficient choice of stepsize makes the discretization formulae free of extra interpolation terms, which can arise from an appropriate approximation of the delayed term.

$$Y(n+2) - \tfrac{1+ah}{1-ah} Y(n) - \tfrac{2bh}{1-ah} Y(n-k+1) = 0, \qquad n = 0, 1, \ldots \qquad \text{(NS)}$$

Theorem

Let $k \geq 2$ be even. Then (NS) *is asymptotically stable if and only if one of the following conditions holds*

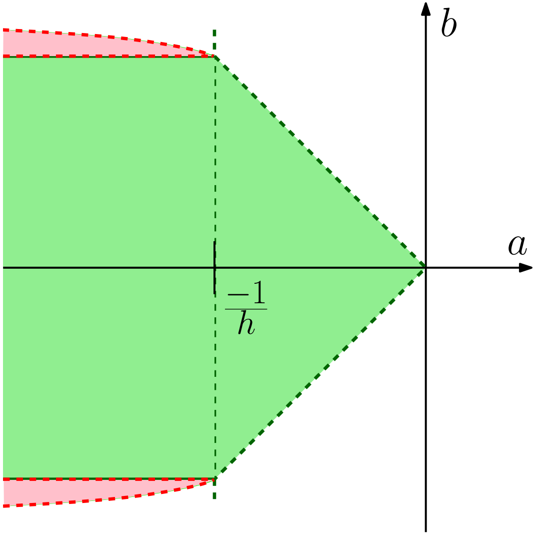
$$|bh| \leq 1, \qquad |b| + a < 0, \tag{6}$$

$$2 < 2b^2h^2 < 1 - ah, \qquad \tau < \tau_1^*(h). \tag{7}$$

where

$$\tau_1^*(h) = h + 2h \arcsin \frac{a + b^2 h}{(1+ah)|b|} \Big/ \arccos \frac{1 + a^2h^2 - 2b^2h^2}{a^2h^2 - 1}.$$

Asymptotic stability region in the case of k even



$$Y(n+2) - \frac{1+ah}{1-ah}Y(n) - \frac{2bh}{1-ah}Y(n-k+1) = 0, \qquad n = 0, 1, \ldots \qquad \text{(NS)}$$

Theorem

Let $k \geq 3$ be odd and $m = (k-1)/2$. Then (NS) is asymptotically stable if and only if one of the following conditions holds

$$a \leq b < -a, \qquad |bh| < 1, \tag{8}$$

$$|b| + a < 0, \qquad (-1)^m bh = 1, \tag{9}$$

$$b + |a| < 0, \qquad bh > -1, \qquad \tau < \tau_2^*(h), \tag{10}$$

$$(-1)^m b + a < 0, \qquad (-1)^m bh > 1, \qquad \tau < \tau_2^*(h), \tag{11}$$

$$(-1)^m b + a > 0, \qquad (-1)^{m+1} bh > 1, \qquad \tau < \tau_2^*(h). \tag{12}$$

where

$$\tau_2^*(h) = h + 2h \arccos \frac{a + b^2 h}{|(1+ah)b|} \Big/ \arccos \frac{1 + a^2h^2 - 2b^2h^2}{|a^2h^2 - 1|}.$$

Asymptotic stability region in the case of k odd and $m = \frac{k-1}{2}$ even

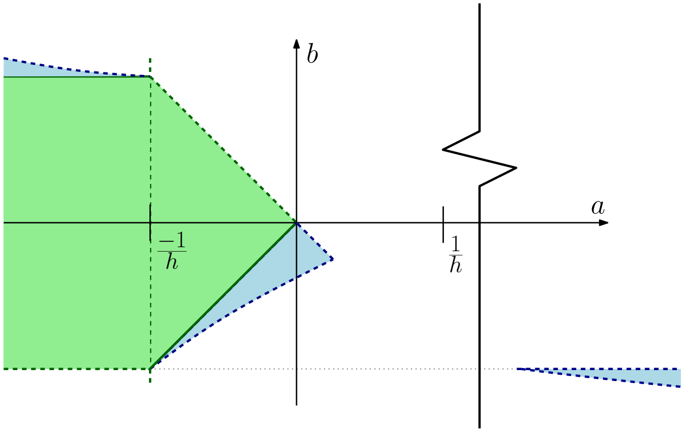


Figure: k odd and $m = \frac{k-1}{2}$ even

Asymptotic stability region in the case of k odd and $m = \frac{k-1}{2}$ odd

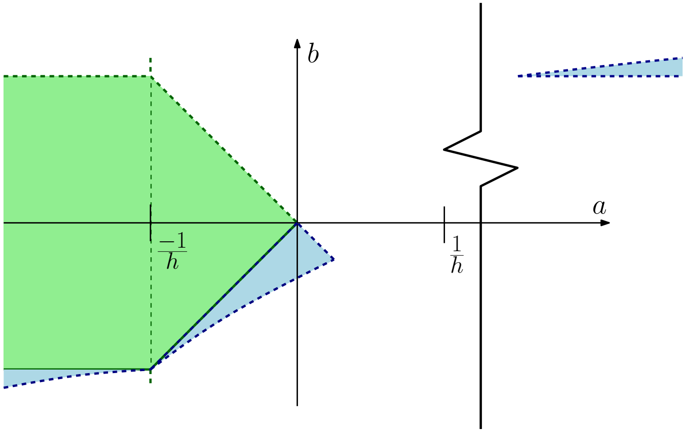


Figure: k odd and $m = \frac{k-1}{2}$ odd

We consider the initial value problem for (DDE) with $\tau = 1$

$$y'(t) = ay(t) + by(t-1)\ , \qquad t > 0\ , \tag{13}$$

$$y(t) = 1 \qquad \text{for} \qquad t \in [-1, 0] \tag{14}$$

and we decide to use formula (NS) to obtain numerical solution.

Example

We consider the initial value problem for (DDE) with $\tau = 1$

$$y'(t) = 30y(t) + \frac{-51}{10}y(t-1)\,, \qquad t \geq 0\,, \tag{15}$$

$$y(t) = 1 \qquad \text{for} \qquad t \in [-1, 0]\,. \tag{16}$$

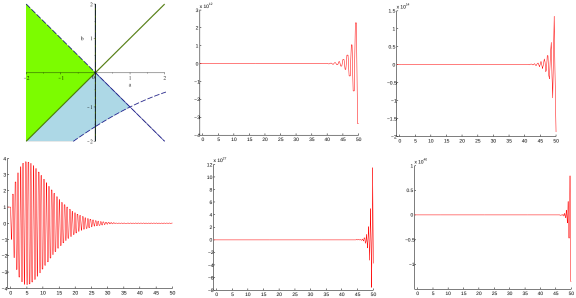


Figure: AS DDE, numerical solutions for $k \in \{3, 4, 5, 6, 7\}$

Example

We consider the initial value problem for (DDE) with $\tau = 1$

$$y'(t) = 30y(t) + \frac{-51}{10}y(t-1)\,, \qquad t \geq 0\,, \tag{17}$$

$$y(t) = 1 \quad \text{for} \quad t \in [-1, 0]\,. \tag{18}$$

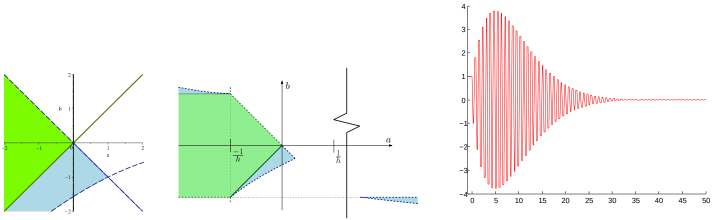


Figure: $(a, b) = (30, -5.1)$, $k = 5$

Example

We consider the initial value problem for (DDE) with $\tau = 1$

$$y'(t) = -y(t) - \frac{3}{2} y(t-1) \, , \qquad t \geq 0 \, , \tag{19}$$

$$y(t) = 1 \qquad \text{for} \qquad t \in [-1, 0] \, . \tag{20}$$

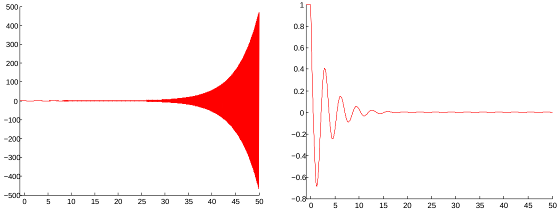


Figure: $(a, b) = (-1, -1.5)$, $k = 50$; $k = 51$

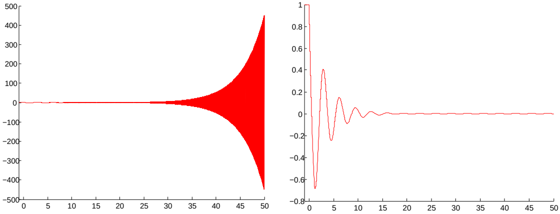
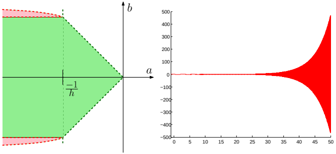
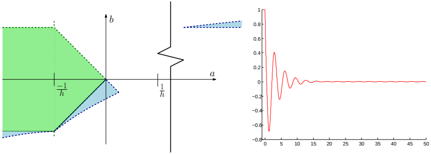


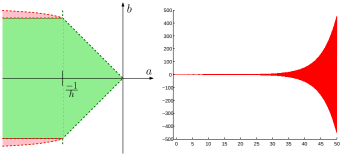
Figure: $(a, b) = (-1, -1.5)$, $k = 52$; $k = 53$



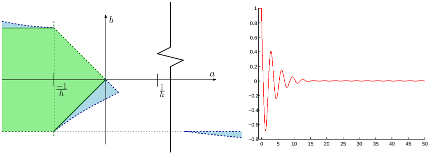
$k = 50$



$k = 51, m = 25$



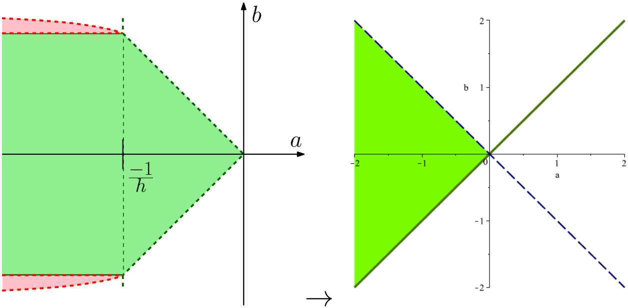
$k = 52$



$k = 53, , m = 26$

What happens if $h \to 0^+$?

In the case of k **even** the asymptotic stability region of (NS) becomes $|b| + a < 0$. With the exception of the boundary, this region corresponds to (1).

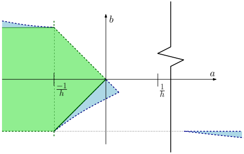


What happens if $h \to 0^+$?

In the case of k **odd** it may be shown (by the L'Hospital rule) that the asymptotic stability conditions turn to

$$a \le b < -a,$$
$$|a| + b < 0, \qquad \tau < \frac{\arccos(-a/b)}{(b^2-a^2)^{1/2}}$$

as $h \to 0^+$. These are equivalent to the conditions defining the asymptotic stability region of (DDE).



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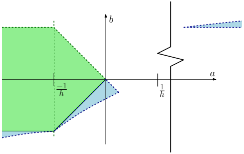

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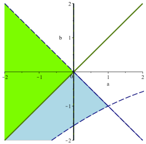

Summary

- Formulation of necessary and sufficient conditions for numerical scheme applied to (DDE)
- Numerical examples - explanation for "unexpected" computation results.

The results are based on ...

- Čermák J, Jánský J, Kundrát P: On necessary and sufficient conditions for the asymptotic stability of higher order linear difference equations. *J. Difference Equ. Appl.* **10(11)**, (2012), 1781–1800.
- Čermák J, Tomášek P: On delay-dependent stability conditions for a three-term linear difference equation. *Funkcial. Ekvac.* **57(1)**, (2014), 91–106.
- Hrabalová, J.; Tomášek P: On stability regions of the modified midpoint method for a linear delay differential equation,*Adv. Differ. Equ.*, **2013(177)**, (2013), 1–10.

- MATLAB and MAPLE computations

THE END