Integrability of natural Hamiltonian systems in 2D curved spaces

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Introduction

- Let $H : M \to \mathbb{R}$ be a smooth scalar called Hamiltonian, and

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{q} = \frac{\partial H}{\partial \boldsymbol{p}}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{p} = -\frac{\partial H}{\partial \boldsymbol{q}}, \tag{1}$$

  the associated equations of motion.
- Introducing $\boldsymbol{x} = (\boldsymbol{q}, \boldsymbol{p})^T$, we can rewite (1) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x} = \boldsymbol{v}_H(\boldsymbol{x}), \quad \boldsymbol{v}_H(\boldsymbol{x}) = \mathbb{I}_n \nabla_{\boldsymbol{x}} H, \quad \mathbb{I}_n = \begin{pmatrix} 0 & \mathbb{E} \\ -\mathbb{E} & 0 \end{pmatrix}. \tag{2}$$

- **Question: How to find all solutions?**

$$\boldsymbol{x}(t) = \varphi(t, \boldsymbol{x}_0), \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0.$$

First integrals help

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x} = \boldsymbol{v}_H(\boldsymbol{x}), \quad \boldsymbol{v}_H(\boldsymbol{x}) = \boldsymbol{I}_{2n}\nabla_{\boldsymbol{x}} H, \tag{3}$$

Definition

A non-constant function $F(\boldsymbol{x}) : M \to \mathbb{R}$ is called a first integral of (3) if $F(\boldsymbol{x}(t)) = \text{const}$ for all solutions $\boldsymbol{x}(t)$.

$$\frac{\mathrm{d}}{\mathrm{d}t}F(\boldsymbol{x}) = \left(\frac{\partial F}{\partial \boldsymbol{x}}\right)^T \boldsymbol{v}_H(\boldsymbol{x}) = \{F, H\}(\boldsymbol{x}) = 0.$$

Theorem (Liouville)

If the Hamiltonian system with $n-$ d.o.f. has n functionally independent first integrals which commute, i.e., $\{F_i, F_j\} = 0$, for every $i, j = 1, \ldots, n$, then the equations of motion are integrable by qudrature.

Question: How to hunt for first integrals?

A particular solution and variational equations help!

- Let $H : \mathbb{C}^{2n} \to \mathbb{C}$ be a holomorphic Hamiltonian, and

$$\frac{\mathrm{d}}{\mathrm{d}t} = \boldsymbol{v}_H(\boldsymbol{x}), \quad \boldsymbol{v}_H(\boldsymbol{x}) = \boldsymbol{I}_{2n}\nabla_{\boldsymbol{x}} H, \qquad \boldsymbol{x} \in \mathbb{C}^{2n}, \quad t \in \mathbb{C}, \tag{4}$$

  the associated Hamilton equations.
- Let $t \to \boldsymbol{\varphi}(t) \in \mathbb{C}^{2n}$ be a non-equilibrium solution of (4).
- The maximal analytic continuation of $\boldsymbol{\varphi}(t)$ defines a Riemann surface $\boldsymbol{\Gamma}$ with t as a local coordinate.

$$\boldsymbol{\Gamma} := \{\boldsymbol{x} \in \mathbb{C}^{2n} | \boldsymbol{x} = \boldsymbol{\varphi}(t),\ t \in U \in \mathbb{C}\}.$$

- Variational equations along $\boldsymbol{\varphi}(t)$ have the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\xi} = \boldsymbol{A}(t) \cdot \boldsymbol{\xi}, \qquad \boldsymbol{A}(t) = \frac{\partial \boldsymbol{v}_H}{\partial \boldsymbol{x}}(\boldsymbol{\varphi}(t)). \tag{5}$$

- We can attach to the equation (5) the differential Galois group $\mathcal{G}$.

Morales-Ramis theorem

Theorem

Assume that a Hamiltonian system is meromorphically integrable in the Liouville sense in a neighbourhood of the analytic phase curve Γ*. Then the identity component of the differential Galois group of the variational equations along* Γ *is Abelian.*

- Morales Ruiz, J. J., *Differential Galois theory and non-integrability of Hamiltonian systems,*
  Volume 179 of *Progress in Mathematics*, Birkhäuser Verlag, Basel, 1999.
- Audin, M., *Les systèmes hamiltoniens et leur intégrabilité,*
  Cours Spécialisés 8, Collection SMF, SMF et EDP Sciences, Paris, 2001.

Aplications of Morales–Ramis theory

- to prove non-integrability of Hamiltonian systems,

A. J. Maciejewski and M. Przybylska, Non-integrability of *ABC* flow, *Phys. Lett. A*, 303(4):265–272, 2002.

T. Stachowiak and W. Szumiński, Non-integrability of constrained double pendula, *Phys. Lett. A*, doi:10.1016/j.physleta.2015.09.052.

Maria Przybylska, Wojciech Szumiński, Non-integrability of flail triple pendulum, *Chaos, Solitons & Fractals*, Vol. 53, August 2013.

- to detection possible integrable cases for Hamiltonian systems depending on parameters.

A. J. Maciejewski, M. Przybylska and H. Yoshida, Necessary conditions for the existence of additional first integrals for Hamiltonian systems with homogeneous potential, *Nonlinearity*, Vol. 25, no 2, s. 255–277, 2012.

W. Szumiński, A. J. Maciejewski and M. Przybylska, Note on integrability of certain homogeneous Hamiltonian systems, *Phys. Lett. A*, Vol. 379, no. 45–46, p. 2970–2976, 2015

Main steps during applications

- Find a particular solution different from equilibrium points,
- calculate VE and NVE,
- check if G^0 is Abelian (most difficult step): we try to transform NVE into the equation with known differential Galois group:
  - Riemann P equation,
  - Lamé equation,
  - an equation of the second order with rational coefficients.

Kovacic, J. An algorithm for solving second order linear homogeneous differential equations. *J. Symbolic Comput.*, 2(1):3–43,

Integrability of homogeneous Hamiltonian equations

Integrability of Hamiltonian systems given by

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + V(\boldsymbol{q}), \qquad (\boldsymbol{q},\boldsymbol{p}) \in \mathbb{C}^{2n},$$

V — homogeneous of degree $k \in \mathbb{Z}$

$$V(\lambda q_1, \dots, \lambda q_n) = \lambda^k V(q_1, \dots, q_n)$$

Definition (standard)

Darboux point $\boldsymbol{d} \in \mathbb{C}^n$ is a non-zero solution of

$$V'(\boldsymbol{d}) = \boldsymbol{d}$$

Particular solution

$$\boldsymbol{q}(t) = \varphi(t)\boldsymbol{d}, \;\; \boldsymbol{p}(t) = \dot{\varphi}(t)\boldsymbol{d} \quad \text{provided} \quad \ddot{\varphi} = -\varphi^{k-1}.$$

Integrability of homogeneous Hamiltonian equations

On the energy level:

$$H(\varphi(t)\boldsymbol{d}, \dot{\varphi}(t)\boldsymbol{d}) = e \in \mathbb{C}^{\star},$$

hyperelliptic curve

$$\dot{\varphi}^2 = \frac{2}{k}\left(\varepsilon - \varphi^k\right), \quad \varepsilon = ke \in \mathbb{C}^{*}.$$

The variational equations

$$\ddot{x} = -\lambda\varphi(t)^{k-2}x, \tag{6}$$

where λ is an eigenvalue of $V''(\boldsymbol{d})$.

Morales Ruiz, J. J., Differential Galois theory and non-integrability of Hamiltonian systems, volume 179 of *Progress in Mathematics*, Birkhäuser Verlag, Basel, 1999.

What is analog of homogeneous systems in curved spaces?

No obvious answer

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + V(\boldsymbol{q}), \qquad (\boldsymbol{q}, \boldsymbol{p}) \in \mathbb{C}^{2n},$$

Our first proposition

$$H = \frac{1}{2} r^{m-k}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) + r^m U(\varphi),$$

where m and k are integers, and $k \neq 0$.

- We obtain obstructions on values of the quantities[1]

$$\lambda = 1 + \frac{U''(\varphi_0)}{kU(\varphi_0)}, \qquad \text{where} \qquad U'(\varphi_0) = 0. \tag{7}$$

[1] see Table 1 in W. Szumiński, A. J. Maciejewski, and M. Przybylska. Note on integrability of certain homogeneous Hamiltonian systems. *Phys. Lett. A*, 379(45-46):2970–2976, 2015

$U(\varphi) = -\cos\varphi$. Superintegrable cases

- **Case 1:** $m = 1,\ k = -5.$

$$H = \frac{1}{2} r^6 \left(p_r^2 + \frac{p_\varphi^2}{r^2} \right) - r\cos\varphi,$$

$$F_1 := r^2 p_\varphi^2 \cos(2\varphi) - r^3 p_r p_\varphi \sin(2\varphi) + r^{-1} \sin\varphi \sin(2\varphi),$$

$$F_2 := r^2 p_\varphi^2 \sin(2\varphi) + r^3 p_r p_\varphi \cos(2\varphi) - r^{-1} \sin\varphi \cos(2\varphi).$$

- **Case 2:** $m = -1,\ k = 1.$

$$H = \frac{1}{2} r^{-2} \left(p_r^2 + \frac{p_\varphi^2}{r^2} \right) - r^{-1}\cos\varphi,$$

$$F_1 := r^{-2} p_\varphi^2 \cos(2\varphi) + r^{-1} p_r p_\varphi \sin(2\varphi) + r \sin\varphi \sin(2\varphi),$$

$$F_2 := -r^{-2} p_\varphi^2 \sin(2\varphi) + r^{-1} p_r p_\varphi \cos(2\varphi) + r \sin\varphi \cos(2\varphi).$$

$U(\varphi) = -\cos\varphi$. Super-integrable cases

- **Case 3:** $m = 1,\ k = 1.$

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) - r\cos\varphi,$$

$$F_1 := r^{-1}p_\varphi^2\cos\varphi + p_r p_\varphi\sin\varphi + \frac{1}{2}r^2\sin^2\varphi,$$

$$F_2 := \left(p_r^2 - r^{-2}p_\varphi^2\right)\cos\varphi\sin\varphi + r^{-1}p_r p_\varphi\cos(2\varphi) - r\sin\varphi.$$

- **Case 4:** $m = -1,\ k = -5.$

$$H = \frac{1}{2}r^4\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) - r^{-1}\cos\varphi,$$

$$F_1 := rp_\varphi^2\cos\varphi - r^2 p_r p_\varphi\sin\varphi + \frac{1}{2}r^{-2}\sin^2\varphi,$$

$$F_2 := r^4\left(p_r^2 - r^{-2}p_\varphi^2\right)\cos\varphi\sin\varphi - r^3 p_r p_\varphi\cos(2\varphi) - r^{-1}\sin\varphi.$$

Higher order first integrals

- $m = 3(k+2), \qquad U(\varphi) = \cosh(\sqrt{3}(k+2)\varphi)$

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) r^{2(k+3)} + r^{3(k+2)}\cosh(\sqrt{3}(k+2)\varphi). \tag{8}$$

Cubic first integral

$$F = (k+2)p_\varphi^3 - \frac{3}{4}r^{3+k}U'(\varphi)p_r + \frac{9}{4}(k+2)r^{k-2}U(\varphi)p_\varphi. \tag{9}$$

- $m = 2k, \qquad U(\varphi) = \cosh((k+2)\varphi)$

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) r^2 + r^{(k+2)}\cosh((k+2)\varphi). \tag{10}$$

Quartic first integral

$$F = r^2(k+2)^2 p_r^2 p_\varphi^2 + 2(k+2)r^{k+2}U'(\varphi)p_r p_\varphi + r^{2(k+2)}U'(\varphi)^2. \tag{11}$$

Another analogue in curved spaces

Our second proposition

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{\mathcal{S}_\kappa(r)^2}\right) + \mathcal{S}_\kappa(r)^m U(\varphi), \qquad (12)$$

where $m \in \mathbb{Z}$ and $U(\varphi)$ is a meromorphic function and $\mathcal{S}_\kappa(r)$ is defined by

$$\mathrm{S}_\kappa(r) := \begin{cases} \frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa} r) & \text{for} \quad \kappa > 0, \\ r & \text{for} \quad \kappa = 0, \\ \frac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa} r) & \text{for} \quad \kappa < 0. \end{cases} \qquad (13)$$

- We obtain obstructions on values of the quantities[2]

$$\lambda = 1 + \frac{U''(\varphi_0)}{kU(\varphi_0)}, \qquad \text{where} \qquad U'(\varphi_0) = 0. \qquad (14)$$

[2] see Table 1 in A. J. Maciejewski, W. Szumiński, and M. Przybylska. Note on integrability of certain homogeneous Hamiltonian systems in 2D constant curvature spaces. *Phys. Lett. A*, 381(7):725–732, 2017

Integrable cases and super-integrable cases

- $U(\varphi) = \cos^k \varphi$ and k-arbitrary

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{\mathcal{S}_\kappa(r)^2}\right) + S_\kappa^m(r)\cos^m\varphi, \tag{15}$$

Linear first integral

$$I_\kappa = p_r \sin\varphi + p_\varphi \cos\varphi\sqrt{\kappa}\cot\sqrt{\kappa}r, \qquad \kappa \neq 0. \tag{16}$$

Limit

$$I_0 = \lim_{\kappa\to 0} I_\kappa = p_r \sin\varphi + r^{-1} p_\varphi \cos\varphi, \tag{17}$$

gives the first integral for the case $\kappa = 0$.

- $U(\varphi) = \cos\varphi$, $\kappa = 0$ and $k = 1$, then there exists additional independent first integral quadratic in momenta

$$I_2 = \left(p_r^2 - \frac{p_\varphi^2}{r^2}\right)\cos\varphi\sin\varphi + r^{-1} p_r p_\varphi \cos(2\varphi) - r\sin\varphi. \tag{18}$$

Thus, in this case the system is maximally super-integrable.

Integrable cases and super-integrable cases

- $U(\varphi) = \cos(2\varphi)$ and $k = 2$.

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{\mathrm{S}_\kappa(r)^2}\right) + \mathrm{S}_\kappa(r)^2\cos(2\varphi) \tag{19}$$

quadratic first integral

$$I = \left[p_r^2 - \left(p_\varphi\frac{\mathrm{C}_\kappa(r)}{\mathrm{S}_\kappa(r)}\right)^2\right]U(\varphi) + p_r p_\varphi\frac{\mathrm{C}_\kappa(r)}{\mathrm{S}_\kappa(r)}U'(\varphi) + 2(c_1^2 + c_2^2)\,\mathrm{S}_\kappa(r)^2. \tag{20}$$

What about an arbitrary form of the metric?

- First system

$$H = \frac{1}{2} r^{m-k} \left(p_r^2 + \frac{p_\varphi^2}{r^2} \right) + r^m U(\varphi),$$

- Second system

$$H = \frac{1}{2} \left(p_r^2 + \frac{p_\varphi^2}{\mathcal{S}_\kappa(r)^2} \right) + \mathcal{S}_\kappa(r)^m U(\varphi),$$

  - When $\kappa = 0$, then $M^2 = \mathbb{E}^2$ is a Cartesian plane
  - When $\kappa > 0$, then $M^2 = \mathbb{S}^2$ is a sphere
  - When $\kappa < 0$, then $M^2 = \mathbb{H}^2$ is a hyperbolic plane.

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Our third proposition

$$H = \frac{1}{2} \left(a(r) p_r^2 + b(r) p_\varphi^2 \right) + c(r) \cos\varphi + d(r) \sin\varphi, \tag{21}$$

where $a(r), b(r), c(r)$ and $d(r)$ are meromorphic functions of variable r.

Main integrability theorem. Auxiliary sets

$$\mathcal{M}_1(\mu) := \left\{ \frac{1}{4}(1+4p)\left(1+4p\pm\sqrt{1+8\mu}\right) \;\middle|\; p\in\mathbb{Z} \right\}, \tag{22}$$

$$\mathcal{M}_2(\mu) := \left\{ \left(p+\frac{1}{2}\right)^2 - \left(\mu+\frac{1}{4}\right)^2 + \mu^2 \;\middle|\; p\in\mathbb{Z} \right\}, \tag{23}$$

$$\mathcal{M}_3(\mu) := \left\{ \left(p+\frac{1}{3}\right)^2 - \left(\mu+\frac{1}{4}\right)^2 + \mu^2 \;\middle|\; p\in\mathbb{Z} \right\}, \tag{24}$$

$$\mathcal{M}_4(\mu) := \left\{ \left(p+\frac{1}{4}\right)^2 - \left(\mu+\frac{1}{4}\right)^2 + \mu^2 \;\middle|\; p\in\mathbb{Z} \right\}, \tag{25}$$

$$\mathcal{M}_5(\mu) := \left\{ \left(p+\frac{1}{5}\right)^2 - \left(\mu+\frac{1}{4}\right)^2 + \mu^2 \;\middle|\; p\in\mathbb{Z} \right\}, \tag{26}$$

$$\mathcal{M}_6(\mu) := \left\{ \left(p+\frac{2}{5}\right)^2 - \left(\mu+\frac{1}{4}\right)^2 + \mu^2 \;\middle|\; p\in\mathbb{Z} \right\}. \tag{27}$$

Theorem (Main Theorem)

Assume that $a(r)$, $b(r)$, $c(r)$ and $d(r)$ are meromorphic functions and there exists a point $r_0 \in \mathbb{Z}$ such that

$$b'(r_0) = c'(r_0) = d'(r_0) = 0, \quad b(r_0) \neq 0, \quad \text{and} \quad c(r_0) \neq -\mathrm{i}d(r_0). \tag{28}$$

If the Hamiltonian system defined by the Hamiltonian

$$H = \frac{1}{2}\left(a(r)p_r^2 + b(r)p_\varphi^2\right) + c(r)\cos\varphi + d(r)\sin\varphi, \tag{29}$$

is integrable in the Liouville sense, then the numbers

$$\begin{aligned} \mu :=& \frac{a(r_0)((c(r_0)+\mathrm{i}d(r_0))b''(r_0) - b(r_0)(c''(r_0)+\mathrm{i}d''(r_0))}{b(r_0)^2(c(r_0)+\mathrm{i}d(r_0))}, \\ \lambda :=& \mathrm{i}\frac{a(r_0)(c(r_0)d''(r_0) - d(r_0)c''(r_0))}{b(r_0)(c(r_0)^2+d(r_0)^2)}, \end{aligned} \tag{30}$$

belong to the following table.

Integrability Table

No.	μ	λ
1	$\mathbb{C}$	$\mathcal{M}_1(\mu) \cup \mathcal{M}_2(\mu)$
2	$2\left(q+\frac{1}{2}\right)^2 - \frac{1}{8}$	$\mathbb{C}$
3	$2q^2 + q$	$\mathcal{M}_3(\mu)$
4	$2\left(q+\frac{1}{3}\right)^2 - \frac{1}{8}$	$\bigcup_{i=3}^{6} \mathcal{M}_i(\mu)$
5	$2\left(q+\frac{1}{5}\right)^2 - \frac{1}{8}$	$\mathcal{M}_3(\mu) \cup \mathcal{M}_6(\mu)$
6	$2\left(q+\frac{2}{5}\right)^2 - \frac{1}{8}$	$\mathcal{M}_3(\mu) \cup \mathcal{M}_5(\mu)$

Table: Integrability table. Here $q \in \mathbb{Z}$ and the sets $\mathcal{M}_i(\mu)$ are defined in (42)–(47).

Outline of the proof. Vector field

- The system

$$\dot{r} = \frac{\partial H}{\partial p_r} = a(r)p_r, \qquad \dot{p}_r = -\frac{\partial H}{\partial r} = -\frac{1}{2}\left(a'(r)p_r^2 + b'(r)p_\varphi^2\right) - c'(r)\cos\varphi - d'(r)\sin\varphi,$$
$$\dot{\varphi} = \frac{\partial H}{\partial p_\varphi} = b(r)p_\varphi, \qquad \dot{p}_\varphi = -\frac{\partial H}{\partial \varphi} = c(r)\sin\varphi - d(r)\cos\varphi. \tag{31}$$

- If $b'(r_0) = c'(r_0) = d'(r_0) = 0$, for a certain $r_0 \in \mathbb{C}$, then the system (31) possesses the invariant manifold

$$\mathcal{N} = \left\{(r, p_r, \varphi, p_\varphi) \in \mathbb{C}^4 \mid r = r_0,\ p_r = 0\right\}, \tag{32}$$

and its restriction to $\mathcal{N}$ is given by

$$\dot{r} = \dot{p}_r = 0, \qquad \dot{\varphi} = b(r_0)p_\varphi, \qquad \dot{p}_\varphi = c(r_0)\sin\varphi - d(r_0)\cos\varphi. \tag{33}$$

$$\dot{\varphi}^2 = 2b(r_0)\left\{E - c(r_0)\cos\varphi - d(r_0)\sin\varphi\right\}. \tag{34}$$

Outline of the proof. Variational equations

- Particular solution

$$\boldsymbol{\varphi}(t) = (0, 0, \varphi(t) p_{\varphi}(t)).$$

- The first order variational equations along $\boldsymbol{\varphi}(t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{X} = \boldsymbol{A}(t)\boldsymbol{X}, \qquad \boldsymbol{A}(t) = \frac{\partial \boldsymbol{v}_H(\boldsymbol{x})}{\partial \boldsymbol{x}}(\boldsymbol{\varphi}(t)), \tag{35}$$

where the matrix $\boldsymbol{A}(t)$ has the form

$$\boldsymbol{A}(t) = \begin{bmatrix} 0 & a(r_0) & 0 & 0 \\ (\Xi - E)\dfrac{b''(r_0)}{b(r_0)} - c''(r_0)\cos\varphi - d''(r_0)\sin\varphi & 0 & 0 & 0 \\ 0 & 0 & 0 & b(r_0) \\ 0 & 0 & \Xi & 0 \end{bmatrix}$$

$$\Xi := c(r_0)\cos\varphi + d(r_0)\sin\varphi.$$

$\boldsymbol{X} = [R, P_R, \Phi, P_\Phi]^T$ denotes the variations of $\boldsymbol{x} = [r, p_r, \varphi, p_\varphi]^T$.

Outline of the proof. Rationalization

- The normal part

$$\begin{pmatrix}\dot R\\ \dot P_R\end{pmatrix}=\begin{pmatrix}0 & a(r_0)\\ (\Xi-E)\,\frac{b''(r_0)}{b(r_0)}-c''(r_0)\cos\varphi-d''(r_0)\sin\varphi & 0\end{pmatrix}\begin{pmatrix}R\\ P_R\end{pmatrix}\qquad(36)$$

can be rewritten as a one second-order differential equation

$$\ddot R=a(r_0)\left(\left(c(r_0)\cos\varphi+d(r_0)-E\sin\varphi\right)\frac{b''(r_0)}{b(r_0)}-c''(r_0)\cos\varphi-d''(r_0)\sin\varphi\right)R.$$

- Change of independent variable

$$t\longrightarrow z:=e^{2\mathrm{i}\varphi(t)}\left(1-\frac{2c(r_0)}{c(r_0)+\mathrm{i}d(r_0)}\right)\qquad(37)$$

on the level $E=0$, transforms NVE into

$$\frac{\mathrm{d}^2R}{\mathrm{d}z^2}+\left(\frac{3}{4z}+\frac{1}{2(z-1)}\right)\frac{\mathrm{d}R}{\mathrm{d}z}-\left(\frac{\mu}{8z^2}+\frac{\lambda}{4z(z-1)}\right)R=0,\qquad(38)$$

Outline of the proof. Rationalization

$$\frac{\mathrm{d}^2R}{\mathrm{d}z^2}+\left(\frac{3}{4z}+\frac{1}{2(z-1)}\right)\frac{\mathrm{d}R}{\mathrm{d}z}-\left(\frac{\mu}{8z^2}+\frac{\lambda}{4z(z-1)}\right)R=0, \tag{39}$$

where

$$\mu:=\frac{a(r_0)((c(r_0)+\mathrm{i}d(r_0))b''(r_0)-b(r_0)(c''(r_0)+\mathrm{i}d''(r_0))}{b(r_0)^2(c(r_0)+\mathrm{i}d(r_0))},$$

$$\lambda:=\mathrm{i}\frac{a(r_0)(c(r_0)d''(r_0)-d(r_0)c''(r_0))}{b(r_0)(c(r_0)^2+d(r_0)^2)},$$

Outline of the proof. Rationalization

$$\frac{\mathrm{d}^2R}{\mathrm{d}z^2}+\left(\frac{3}{4z}+\frac{1}{2(z-1)}\right)\frac{\mathrm{d}R}{\mathrm{d}z}-\left(\frac{\mu}{8z^2}+\frac{\lambda}{4z(z-1)}\right)R=0, \qquad (39)$$

where

$$\mu := \frac{a(r_0)((c(r_0)+\mathrm{i}d(r_0))b''(r_0)-b(r_0)(c''(r_0)+\mathrm{i}d''(r_0))}{b(r_0)^2(c(r_0)+\mathrm{i}d(r_0))},$$

$$\lambda := \mathrm{i}\frac{a(r_0)(c(r_0)d''(r_0)-d(r_0)c''(r_0))}{b(r_0)(c(r_0)^2+d(r_0)^2)},$$

- Form of the Riemman P equation

$$R''+\left(\frac{1-\alpha-\alpha'}{z}+\frac{1-\gamma-\gamma'}{z-1}\right)R'+\left(\frac{\alpha\alpha'}{z^2}+\frac{\gamma\gamma'}{(z-1)^2}+\frac{\beta\beta'-\alpha\alpha'-\gamma\gamma'}{z(z-1)}\right)R=0,$$

The differences of exponents at singularities $z=0, z=1$ and $z=\infty$

$$\rho=\alpha-\alpha'=\frac{\sqrt{\Delta^2-16\lambda}}{4}, \qquad \sigma=\gamma-\gamma'=\frac{1}{2}, \qquad \tau=\beta-\beta'=\frac{\Delta}{4}, \qquad (40)$$

where

$$\Delta=\sqrt{1+16\lambda+8\mu}.$$

Solvability of Riemann P equation. Kimura theorem

Theorem (Kimura)

The identity component of the differential Galois group of the Riemann P equation is solvable iff

A. *at least one of the four numbers* $\rho+\sigma+\tau$, $-\rho+\sigma+\tau$, $\rho-\sigma+\tau$, $\rho+\sigma-\tau$ *is an odd integer, or*

B. *the numbers* ρ *or* $-\rho$ *and* σ *or* $-\sigma$ *and* τ *or* $-\tau$ *belong (in an arbitrary order) to some of appropriate fifteen families forming the so-called Schwarz's table fifteen families*

1	$1/2+l$	$1/2+s$	arbitrary complex number	
2	$1/2+l$	$1/3+s$	$1/3+q$	
3	$2/3+l$	$1/3+s$	$1/3+q$	$l+s+q$ even
4	$1/2+l$	$1/3+s$	$1/4+q$	
5	$2/3+l$	$1/4+s$	$1/4+q$	$l+s+q$ even
6	$1/2+l$	$1/3+s$	$1/5+q$	
7	$2/5+l$	$1/3+s$	$1/3+q$	$l+s+q$ even
8	$2/3+l$	$1/5+s$	$1/5+q$	$l+s+q$ even
9	$1/2+l$	$2/5+s$	$1/5+q$	
10	$3/5+l$	$1/3+s$	$1/5+q$	$l+s+q$ even
11	$2/5+l$	$2/5+s$	$2/5+q$	$l+s+q$ even
12	$2/3+l$	$1/3+s$	$1/5+q$	$l+s+q$ even
13	$4/5+l$	$1/5+s$	$1/5+q$	$l+s+q$ even
14	$1/2+l$	$2/5+s$	$1/3+q$	
15	$3/5+l$	$2/5+s$	$1/3+q$	$l+s+q$ even

where $l, s, q \in \mathbb{Z}$.

Kimura theorem: Condition A

- The case A of the Kimura Theorem is satisfied if and only if one of the numbers

$$\begin{aligned} \rho+\sigma+\tau &= \frac{1}{4}\left(2+\Delta+\sqrt{\Delta^2-16\lambda}\right), \\ -\rho+\sigma+\tau &= \frac{1}{4}\left(2+\Delta-\sqrt{\Delta^2-16\lambda}\right), \\ \rho-\sigma+\tau &= \frac{1}{4}\left(-2+\Delta+\sqrt{\Delta^2-16\lambda}\right), \\ \rho+\sigma-\tau &= \frac{1}{4}\left(2-\Delta+\sqrt{\Delta^2-16\lambda}\right) \end{aligned}$$

is an odd integer. It is easy to check that if one the above numbers is an odd integer, then $\lambda \in \mathcal{M}_1(\mu)$, where

$$\mathcal{M}_1(\mu) = \left\{ \frac{1}{4}(1+4p)\left(1+4p \pm \sqrt{\Delta^2-16\lambda}\right) \;\middle|\; p \in \mathbb{Z} \right\}$$

Kimura Theorem: Condition B

In this case the quantities ρ or $-\rho$, σ or $-\sigma$ and τ or $-\tau$ must belong to Schwarz's table. As $\sigma = \frac{1}{2}$ only items 1, 2, 4, 6, 9, or 14 are allowed.

Case 1.

- $\pm\rho = 1/2 + q$, for a certain $q \in \mathbb{Z}$, then $\mu = 2\left(q + \frac{1}{2}\right)^2 - \frac{1}{8}$. In this case τ is arbitrary, and thus λ is arbitrary.
- $\pm\tau = 1/2 + p$, for certain $p \in \mathbb{Z}$, then $\lambda \in \mathcal{M}_2(\mu)$. In this case ρ is arbitrary, so μ is arbitrary.

Case 2. In this case $\pm\rho = 1/3 + q$ and $\pm\tau = 1/3 + p$, for certain $q, p \in \mathbb{Z}$. These conditions imply that $\lambda \in \mathcal{M}_3(\mu)$, and

$$\mu = 2\left(q + \frac{1}{3}\right)^2 - \frac{1}{8}. \tag{41}$$

Case 4.

- $\pm\rho = 1/3 + q$, and $\pm\tau = 1/4 + p$, for certain $q, p \in \mathbb{Z}$, then $\lambda \in \mathcal{M}_4(\mu)$ and μ is given by (41).
- $\pm\rho = 1/4 + q$, and $\pm\tau = 1/3 + p$, for certain $q, p \in \mathbb{Z}$, then $\lambda \in \mathcal{M}_3(\mu)$ and $\mu = 2q^2 + q$.

Integrability Table

No.	μ	λ
1	$\mathbb{C}$	$\mathcal{M}_1(\mu) \cup \mathcal{M}_2(\mu)$
2	$2\left(q+\frac{1}{2}\right)^2 - \frac{1}{8}$	$\mathbb{C}$
3	$2q^2 + q$	$\mathcal{M}_3(\mu)$
4	$2\left(q+\frac{1}{3}\right)^2 - \frac{1}{8}$	$\bigcup_{i=3}^{6} \mathcal{M}_i(\mu)$
5	$2\left(q+\frac{1}{5}\right)^2 - \frac{1}{8}$	$\mathcal{M}_3(\mu) \cup \mathcal{M}_6(\mu)$
6	$2\left(q+\frac{2}{5}\right)^2 - \frac{1}{8}$	$\mathcal{M}_3(\mu) \cup \mathcal{M}_5(\mu)$

Table: Integrability table. Here $q \in \mathbb{Z}$ and the sets $\mathcal{M}_i(\mu)$ are defined in (42)–(47).

Main integrability theorem. Auxiliary sets

$$\mathcal{M}_1(\mu) := \left\{ \frac{1}{4}(1+4p)\left(1+4p \pm \sqrt{1+8\mu}\right) \;\middle|\; p \in \mathbb{Z} \right\}, \tag{42}$$

$$\mathcal{M}_2(\mu) := \left\{ \left(p+\frac{1}{2}\right)^2 - \left(\mu+\frac{1}{4}\right)^2 + \mu^2 \;\middle|\; p \in \mathbb{Z} \right\}, \tag{43}$$

$$\mathcal{M}_3(\mu) := \left\{ \left(p+\frac{1}{3}\right)^2 - \left(\mu+\frac{1}{4}\right)^2 + \mu^2 \;\middle|\; p \in \mathbb{Z} \right\}, \tag{44}$$

$$\mathcal{M}_4(\mu) := \left\{ \left(p+\frac{1}{4}\right)^2 - \left(\mu+\frac{1}{4}\right)^2 + \mu^2 \;\middle|\; p \in \mathbb{Z} \right\}, \tag{45}$$

$$\mathcal{M}_5(\mu) := \left\{ \left(p+\frac{1}{5}\right)^2 - \left(\mu+\frac{1}{4}\right)^2 + \mu^2 \;\middle|\; p \in \mathbb{Z} \right\}, \tag{46}$$

$$\mathcal{M}_6(\mu) := \left\{ \left(p+\frac{2}{5}\right)^2 - \left(\mu+\frac{1}{4}\right)^2 + \mu^2 \;\middle|\; p \in \mathbb{Z} \right\}. \tag{47}$$

Theorem (Main Theorem)

Assume that $a(r)$, $b(r)$, $c(r)$ and $d(r)$ are meromorphic functions and there exists a point $r_0 \in \mathbb{Z}$ such that

$$b'(r_0) = c'(r_0) = d'(r_0) = 0, \quad b(r_0) \neq 0, \quad \text{and} \quad c(r_0) \neq -\mathrm{i}d(r_0). \tag{48}$$

If the Hamiltonian system defined by the Hamiltonian

$$H = \frac{1}{2}\left(a(r)p_r^2 + b(r)p_\varphi^2\right) + c(r)\cos\varphi + d(r)\sin\varphi, \tag{49}$$

is integrable in the Liouville sense, then the numbers

$$\begin{aligned} \mu :=& \frac{a(r_0)((c(r_0)+\mathrm{i}d(r_0))b''(r_0) - b(r_0)(c''(r_0)+\mathrm{i}d''(r_0))}{b(r_0)^2(c(r_0)+\mathrm{i}d(r_0))}, \\ \lambda :=& \mathrm{i}\frac{a(r_0)(c(r_0)d''(r_0) - d(r_0)c''(r_0))}{b(r_0)(c(r_0)^2+d(r_0)^2)}, \end{aligned} \tag{50}$$

belong to the Table 2.

Application of the Theorem 2. First example

- Let us consider the following Hamiltonian function

$$H = \frac{1}{2}\left\{p_r^2 + \left(n + \sin^{-2} r\right) p_\varphi^2\right\} + \sin r \cos\varphi, \tag{51}$$

with $n \in \mathbb{Z}$.
The functions a, b, c, d are

$$a(r) = 1, \qquad b(r) = n + \sin^{-2} r, \qquad c(r) = \sin r, \qquad d(r) = 0. \tag{52}$$

- We take a point $r_0 = \pi/2$, at which the condition (48) is fulfilled.
- The values of μ and λ at r_0 are given by

$$\mu = \frac{3+n}{(1+n)^2}, \qquad \lambda = 0. \tag{53}$$

- Possibly integrable cases $n \in \{0, 1, 3, -3, -2, 11\}$.

First example. Integrable cases

1. For $n = 0$, the system (51) possesses linear first integral

$$F = p_r \sin\varphi + p_\varphi \cot r \cos\varphi. \tag{54}$$

2. For $n = 1$, the Hamiltonian (51) coincide with the famous Kovalevskaya case defined on sphere $\mathbb{S}^2$ that has the quartic first integral

$$\begin{aligned} I =& p_\varphi^4 \sin^{-2} r + p_r^2 p_\varphi^2 + 2p_\varphi^2 \sin^{-1} r \cos\varphi + 2p_r p_\varphi \cos r \sin\varphi \\ &+ \frac{1}{4}\left(\cos(2\varphi) + 2\cos(2r)\sin^2\varphi\right), \end{aligned} \tag{55}$$

3. For $n = 3$, the Hamiltonian (51) corresponds to the Goryachiev–Chaplygin system defined on sphere $\mathbb{S}^2$ that posses the following first integrals cubic in momenta

$$I = p_\varphi^3 \cot^2 r + p_\varphi p_r^2 + p_r \cos r \sin\varphi + p_\varphi \frac{\cos^2 r}{\sin r}\cos\varphi, \tag{56}$$

...and what about the cases $n \in \{-3, -2, 11\}$?

First example. Not integrable cases

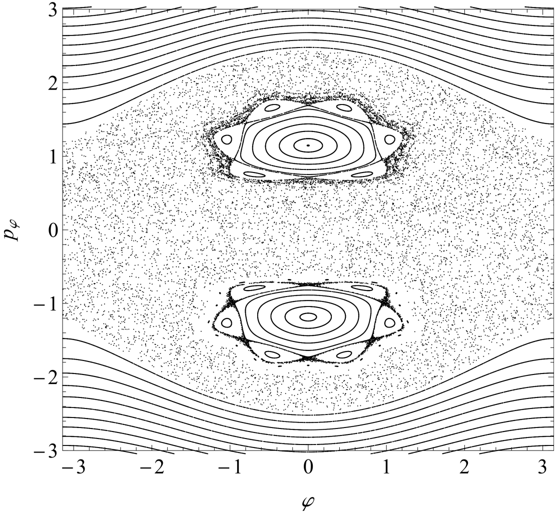

Figure: Poincaré section for $n = -3$ on the level $E = 2$. Cross plane $r = \pi/2$, $p_r > 0$

First example. Not integrable cases

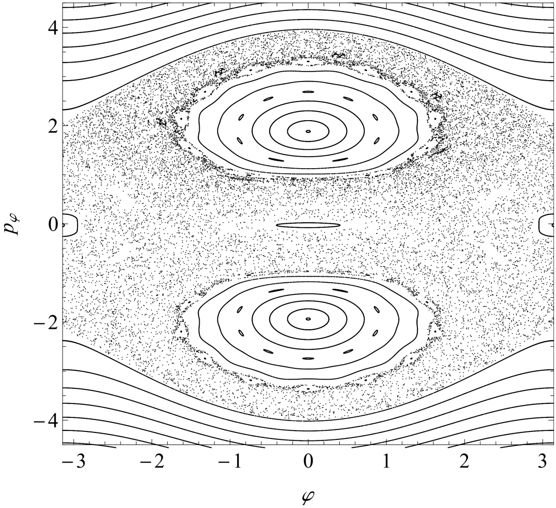

Figure: Poincaré section for $n = -2$ on the level $E = 2$. Cross plane $r = \pi/2$, $p_r > 0$

First example. Not integrable cases

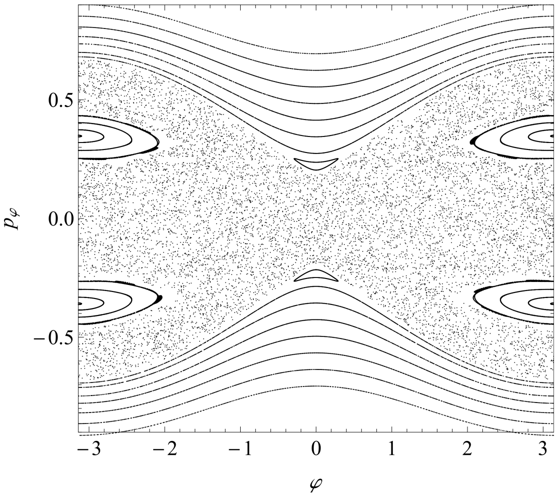

Figure: Poincaré section for $n = 11$ on the level $E = 2$. Cross plane $r = \pi/2$, $p_r > 0$

Application of the Theorem 2. Second example

- Let us consider the following Hamiltonian function

$$H = \frac{1}{2}\left\{p_r^2 + \left(n^2 + k^2\sin^{-2} r + \frac{n^2}{4}\tan^2 r\right)p_\varphi^2\right\} + \sin^k r\cos^{\frac{n}{2}} r\cos\varphi, \qquad (57)$$

where $k, n \in \mathbb{Z}$.

- The functions a, b, c, d are

$$a(r) = 1, \qquad b(r) = n^2 + k^2\sin^{-2} r + \frac{n^2}{4}\tan^2 r, \qquad c(r) = \sin^k r\cos^{\frac{n}{2}} r, \qquad d(r) = 0.$$

- We take a point $r_0 = \operatorname{arccot}\left(\sqrt{n/(2k)}\right)$, at which $b'(r_0) = c'(r_0) = 0$.
- The values of μ and λ at r_0 are given by

$$\mu = \frac{(2k+n)(k^2 + n(n+2) + k(n+4))}{(k^2 + kn + n^2)^2}, \qquad \lambda = 0. \qquad (58)$$

- Possibly integrable cases

1. $n = -4$, and $k \in \{4, 8\}$,
2. $n = -2$, and $k \in \{0, -2\}$,
3. $n = -1$, and $k \in \{\pm 1, 2\}$,
4. $n = 0$, and $k \in \{\pm 1, \pm 2, \pm 4, 8, 24\}$,
5. $n = 1$, and $k \in \{0, \pm 1, -2\}$,
6. $n = 2$, and $k \in \{0, \pm 2, \pm 4\}$,
7. $n = 4$, and $k \in \{0, 4\}$,

Second example. Integrable cases

1. For $n = 0,\ k = -2$ the system (57) is separable with the first integral

$$I = \frac{1}{2}p_\varphi^2 + \cos(2\varphi). \tag{59}$$

2. The values $n = 0,\ k = 1$ correspond to case given in 54.
3. For $n = 0,\ k = 2$ the system (57) posses a quadratic first integral

$$I = \left(p_r^2 - 4p_\varphi^2 \cot^2 r\right)\cos(\varphi) - 4p_r p_\varphi \cot r \sin(\varphi) - \cos(2r). \tag{60}$$

4. For $n = 2, k = 0$ the Hamiltonian (57) corresponds to the integrable Goryachiev–Chaplygin system with the first integral given in (56).
5. For $n = -1,\ k = 1$ the Hamiltonian (57) has a cubic first integral

$$I = \left(4\sin^{-2} r + \tan^2 r\right) p_\varphi^3 + 4p_r^2 p_\varphi + 8p_r\sqrt{\cos r}\sin\varphi + \frac{2(3+\cos(2r))}{\sqrt{\cos r}\sin r} p_\varphi \cos\varphi.$$

Dullin and Matveev[3]

[3] H. R. Dullin and V. S. Matveev. A new integrable system on the sphere. *Math. Res. Lett.*, 11(5-6):715–722, 2004.

Second example. Not integrable cases

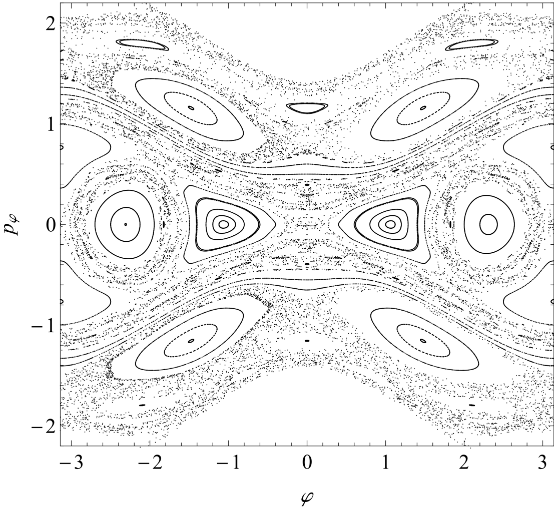

Figure: Poincaré section with $n = 1$, $k = 0$ at the level $E = 3$ on the surface $r = 1$

Second example. Not integrable cases

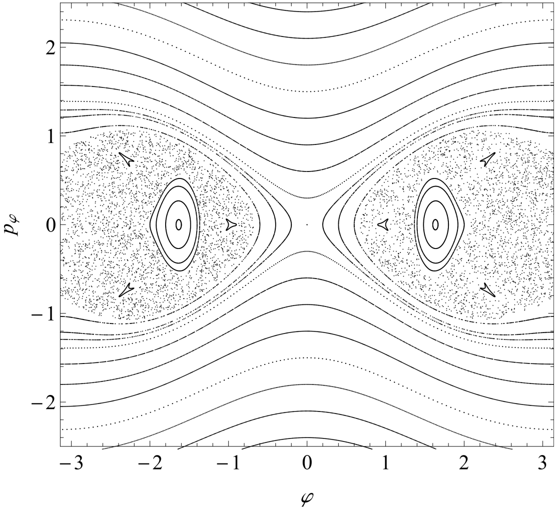

Figure: Poincaré section with $n = 1, k = -1$ at the level $E = 3$ on the surface $r = 1$

Second example. Not integrable cases

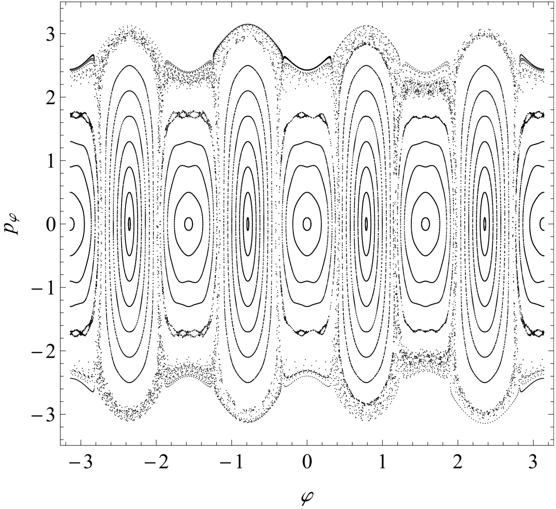

Figure: Poincaré section with $n = 0$, $k = 4$ at the level $E = 3$ on the surface $r = 1$

Summary

Conclusions

- Morales–Ramis theory - the most effective method
- New integrable as well as super-integrable cases were detected

Questions and open problems

- If the necessary integrability conditions are satisfied but it seems that the system is chaotic, how to proof its non-integrability?
- To apply the Main Theorem to the Hamiltonian

$$H = \frac{1}{2}\left(a(r)p_r^2 + b(r)p_\varphi^2\right) + c(r)\cos\varphi + d(r)\sin\varphi, \tag{62}$$

with a more complex form of functions a, b, c, d, and to find new, still unknown integrable cases.

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THANK YOU FOR YOUR ATTENTION!

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