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SOME REMARKS ON FUNCTIONAL DIFFERENTIAL EQUATIONS IN ABSTRACT SPACES

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1. Statement of problem

On the interval $[a,b]$ we consider the functional differential equation

$$\boxed{v'(t) = G(v)(t)} \tag{AE}$$

in the Banach space $\langle \mathbb{X}, \|\cdot\|_{\mathbb{X}} \rangle$, where

- $G\colon C([a,b];\mathbb{X}) \to B([a,b];\mathbb{X})$ is a continuous operator,
  - ▷ $C([a,b];\mathbb{X})$ is the Banach space of continuous abstract functions $v\colon [a,b] \to \mathbb{X}$ endowed with the norm
  $$\|v\|_{C([a,b];\mathbb{X})} = \max\{\|v(t)\|_{\mathbb{X}} : t \in [a,b]\},$$
  - ▷ $B([a,b];\mathbb{X})$ is the Banach space of Bochner integrable abstract functions $g\colon [a,b] \to \mathbb{X}$ endowed with the norm
  $$\|g\|_{B([a,b];\mathbb{X})} = \int_a^b \|g(s)\|_{\mathbb{X}} \mathrm{d}s,$$
- G satisfies the (local) Carathéodory condition, i. e., for any $r > 0$ there exists $q_r \in L([a,b];\mathbb{R})$ such that
$$\|G(w)(t)\|_{\mathbb{X}} \leq q_r(t) \quad \text{for a. e. } t \in [a,b] \text{ and all } w \in C([a,b];\mathbb{X}),\ \|w\|_{C([a,b];\mathbb{X})} \leq r.$$

Definition 1. By a *solution* of equation (AE) we understand an abstract function $v\colon [a,b] \to \mathbb{X}$ which is strongly absolutely continuous on $[a,b]$, differentiable a. e. on $[a,b]$, and satisfies equality (AE) a. e. on $[a,b]$.

Remark 2. In Definition 1:

(a) The function $v\colon [a,b] \to \mathbb{X}$ is strongly absolutely continuous on $[a,b]$ – $v \in AC([a,b];\mathbb{X})$, i. e., for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any system $\{[a_k, b_k]\}_{k=1}^n$ of mutually non-overlapping subintervals of $[a,b]$, the implication
$$\sum_{k=1}^{n} (b_k - a_k) < \delta \quad \Rightarrow \quad \sum_{k=1}^{n} \|v(b_k) - v(a_k)\|_{\mathbb{X}} < \varepsilon$$
holds.

(b) Differentiability a. e. on $[a,b]$ has to be assumed – it does not follow from strong absolute continuity (in general). Indeed, let $\mathbb{X} = L([0,1];\mathbb{R})$ and
$$v(t)(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq t \leq 1, \\ 0 & \text{if } 0 \leq t < x \leq 1. \end{cases}$$
Then v is strongly absolutely continuous on $[0,1]$, but not differentiable a. e. on $[0,1]$ (see [7, Example 7.3.9]).

(c) Solutions of equation (AE) are understood as global and strong ones, notions like local existence and extendability of solutions have no sense.

Remark 3. Equation (AE) differs from frequently studied abstract differential equations of the type

$$v' = A(t)v + f(t, v_t),$$

where $A(t)$ are usually densely closed linear operators with values in $\mathbb{X}$ that generate a semigroup .... In those cases so-called mild solutions are usually considered, i. e., solutions of the corresponding integral equation

$$v(t) = \widehat{V}(t,0)v(0) + \int_0^t \widehat{V}(t,s)f(s,v_s)\mathrm{d}s,$$

where $\widehat{V}(t,s)$ denotes an evolution operator for $A(t)$.

Particular cases of (AE)

We mention here two natural and straightforward particular cases of equation (AE):

(A) $\mathbb{X} = \mathbb{R}$ – scalar first-order functional differential equations (FDEs), for instance,

- differential equation with an argument deviation

$$v'(t) = f(t, v(t), v(\tau(t))),$$

  where $f\colon [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ is a Carathéodory function and $\tau\colon [a,b] \to [a,b]$ is a measurable function,

- integro-differential equation

$$v'(t) = \int_a^b K(t,s)v(\tau(s))\mathrm{d}s,$$

  where $K\colon [a,b] \times [a,b] \to \mathbb{R}$ and $\tau\colon [a,b] \to [a,b]$ are suitable functions,

- differential equation with maximum

$$v'(t) = p(t) \max\left\{v(s) : \tau_1(t) \le s \le \tau_2(s)\right\} + q(s),$$

  where $p, q \in L([a,b];\mathbb{R})$ and $\tau_1, \tau_2\colon [a,b] \to [a,b]$ are measurable functions.

(B) $\mathbb{X} = \mathbb{R}^n$ – systems of first-order FDEs and scalar higher-order FDEs

For both cases $\mathbb{R}$ and $\mathbb{R}^n$, we have some results concerning solvability as well as unique solvability of various boundary value problems, theorems on differential inequalities (maximum principles), oscillations, ...

In order to extend our results for FDEs in abstract spaces, some additional operations and structures are needed in $\mathbb{X}$ (like ordering, positivity, monotonicity, unit element, ...).

$$\Downarrow$$

We are interested in some other particular cases of equation (AE) besides (A) and (B)!

2. Hyperbolic functional differential equation

On the rectangle $\mathcal{D} = [a, b] \times [c, d]$ we consider the hyperbolic functional differential equation

$$\boxed{\frac{\partial^2 u(t,x)}{\partial t\,\partial x} = F(u)(t,x),} \tag{HE}$$

where

- $F\colon C(\mathcal{D};\mathbb{R}) \to L(\mathcal{D};\mathbb{R})$ is a continuous operator,
  - ▷ $C(\mathcal{D};\mathbb{R})$ is the Banach space of continuous functions $u\colon \mathcal{D} \to \mathbb{R}$ endowed with the norm $\|u\|_{C(\mathcal{D};\mathbb{R})} = \max\{|u(t,x)| : (t,x) \in \mathcal{D}\}$,
  - ▷ $L(\mathcal{D};\mathbb{R})$ is the Banach space of Lebesgue integrable functions $h\colon \mathcal{D} \to \mathbb{R}$ endowed with the norm $\|h\|_{L(\mathcal{D};\mathbb{R})} = \iint_{\mathcal{D}} |h(s,\eta)| \mathrm{d}s\mathrm{d}\eta$,
- F satisfies the (local) Carathéodory condition, i. e., for any $r > 0$ there exists $q_r \in L(\mathcal{D};\mathbb{R})$ such that

$$|F(z)(t,x)| \leq q_r(t,x) \quad \text{for a. e. } (t,x) \in \mathcal{D} \text{ and all } z \in C(\mathcal{D};\mathbb{R}),\ \|z\|_{C(\mathcal{D};\mathbb{R})} \leq r.$$

Definition 4. By a *solution* of equation (HE) we understand a function $u\colon \mathcal{D} \to \mathbb{R}$ which is absolutely continuous on $\mathcal{D}$ in the sense of Carathéodory and satisfies equality (HE) a. e. on $\mathcal{D}$.

2.1. Absolute continuity in the sense of Carathéodory

Several notions of absolute continuity of functions of two variables can be found in the existing literature. Let us mention, for instance, absolute continuity in the sense of Schwart, Banach, or Tonelli and 2-absolute continuity introduced by Malý. However, for a meaningful definition of strong solutions of equation (HE), absolute continuity in the sense of Carathéodory (see [1]) is the right one. In the recent terminology, the definition reads as follows:

Definition 5. We say that a function $u\colon \mathcal{D} \to \mathbb{R}$ is *absolutely continuous in the sense of Carathéodory* and we write $u \in AC(\mathcal{D};\mathbb{R})$ if the following two conditions hold:

(a) the function of rectangles

$$\Phi_u([t_1,t_2] \times [x_1,x_2]) = u(t_1,x_1) - u(t_1,x_2) - u(t_2,x_1) + u(t_2,x_2) \quad \text{for } [t_1,t_2] \times [x_1,x_2] \subseteq \mathcal{D}$$

associated with u is absolutely continuous[1],

(b) the function $u(\cdot,c)\colon [a,b] \to \mathbb{R}$ and $u(a,\cdot)\colon [c,d] \to \mathbb{R}$ are absolutely continuous.

[1] Let $\mathcal{S}(\mathcal{D})$ denote the system of rectangles $[t_1,t_2] \times [x_1,x_2]$ contained in $\mathcal{D}$. A function $F\colon \mathcal{S}(\mathcal{D}) \to \mathbb{R}$ is said to be *absolutely continuous* (see [3, §7.3]) if it is additive and for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any system $\{[a_k,b_k] \times [c_k,d_k]\}_{k=1}^n$ of mutually non-overlapping rectangles contained in $\mathcal{D}$, the implication

$$\sum_{k=1}^{n} (b_k - a_k)(d_k - c_k) < \delta \quad \Rightarrow \quad \sum_{k=1}^{n} \big|F([a_k,b_k] \times [c_k,d_k])\big| < \varepsilon$$

holds.

Carathéodory proved in [1] that the set of those functions coincides with the class of functions admitting a certain integral representation. However, for the study of equation (AE), it is necessary to know that absolutely continuous functions in the sense of Carathéodory can be equivalently characterised in terms of properties with respect to each of their variables.

Notation. In what follows we denote:

$u'_{[1]}(t,x)$ (or $u'_t(t,x)$) –	the first-order partial derivative of the function u at the point (t,x) with respect to the first variable,
$u'_{[2]}(t,x)$ (or $u'_x t,x)$) –	the first-order partial derivative of the function u at the point (t,x) with respect to the second variable,
$u''_{[1,2]}(t,x)$ (or $u''_{tx}(t,x)$) –	the mixed second-order partial derivative of the function u at the point (t,x),
$u''_{[2,1]}(t,x)$ (or $u''_{xt}(t,x)$) –	the mixed second-order partial derivative of the function u at the point (t,x).

Proposition 6 ([8, Theorem 3.1])**.** *The following assertions are equivalent:*

(1) The function $u\colon \mathcal{D}\to\mathbb{R}$ is absolutely continuous in the sense of Caratheéodory.

(2) The function $u\colon \mathcal{D}\to\mathbb{R}$ admits the integral representation

$$u(t,x) = e + \int_a^t f(s)\mathrm{d}s + \int_c^x g(\eta)\mathrm{d}\eta + \iint_{[a,t]\times[c,x]} h(s,\eta)\mathrm{d}s\mathrm{d}\eta \quad \text{for } (t,x)\in\mathcal{D},$$

where $e\in\mathbb{R}$, $f\in L([a,b];\mathbb{R})$, $g\in L([c,d];\mathbb{R})$, and $h\in L(\mathcal{D};\mathbb{R})$.

(3) The function $u\colon \mathcal{D}\to\mathbb{R}$ satisfies the following conditions:

(a) $u(\cdot,x)\in AC([a,b];\mathbb{R})$ for every $x\in[c,d]$,
$u(a,\cdot)\in AC([c,d];\mathbb{R})$,

(b) $u'_{[1]}(t,\cdot)\in AC([c,d];\mathbb{R})$ for almost all $t\in[a,b]$,

(c) $u''_{[1,2]}\in L(\mathcal{D};\mathbb{R})$.

(4) The function $u\colon \mathcal{D}\to\mathbb{R}$ satisfies the following conditions:

(A) $u(t,\cdot)\in AC([c,d];\mathbb{R})$ for every $t\in[a,b]$,
$u(\cdot,c)\in AC([a,b];\mathbb{R})$,

(B) $u'_{[2]}(\cdot,x)\in AC([a,b];\mathbb{R})$ for almost all $x\in[c,d]$,

(C) $u''_{[2,1]}\in L(\mathcal{D};\mathbb{R})$.

2.2. Initial value problems for equation (HE)

Two main initial value problems for equation (HE) are studied in the literature:

Darboux problem

The values of the solution u are prescribed on both characteristics $t = a$ and $x = c$, i. e., the initial conditions are

$$u(t,c) = \alpha(t) \quad \text{for } t \in [a,b], \quad u(a,x) = \beta(x) \quad \text{for } x \in [c,d], \tag{D}$$

where $\alpha \in AC([a,b];\mathbb{R})$, $\beta \in AC([c,d];\mathbb{R})$ are such that $\alpha(a) = \beta(c)$ (see Fig. 1).

By using Proposition 6, one can show that the function u is a solution of the Darboux problem (AE), (D) if and only if it is a solution of the integral equation

$$u(t,x) = -\alpha(a) + \alpha(t) + \beta(x) + \int_a^t \int_c^x F(u)(s,\eta)\mathrm{d}\eta\mathrm{d}s$$

in the space $C(\mathcal{D};\mathbb{R})$.

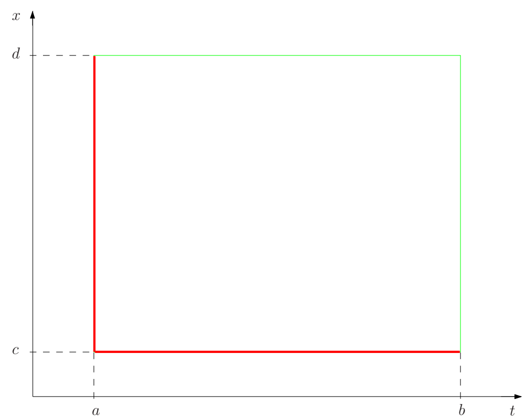


Fig. 1. Darboux problem

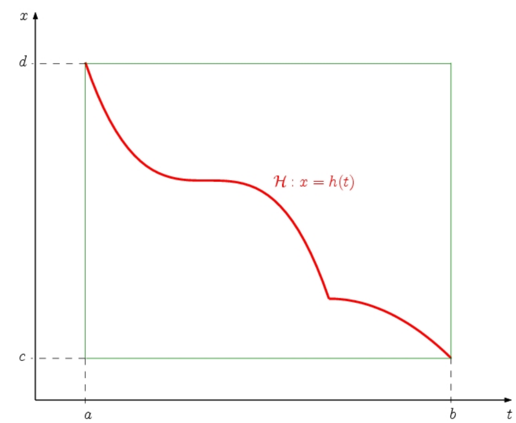


Fig. 2. Cauchy problem

Cauchy problem

Let $\mathcal{H}$ be a curve, which is defined as the graph of a decreasing continuous (not absolutely continuous, in general) function $h\colon [a,b] \to [c,d]$ such that $h(a) = d$ and $h(b) = c$. The values of the solution u and its partial derivative $u'_{[2]}$ are prescribed on $\mathcal{H}$ as follows:

$$u(t,h(t)) = g(t) \quad \text{for } t \in [a,b], \quad u'_{[2]}(h^{-1}(x),x) = \psi(x) \quad \text{for a. e. } x \in [c,d], \tag{C}$$

where $g \in C([a,b];\mathbb{R})$, $\psi \in L([c,d];\mathbb{R})$ are such that the function

$$t \mapsto g(t) + \int_{h(t)}^d \psi(\eta)\mathrm{d}\eta$$

is absolutely continuous on $[a,b]$[2] (see Fig. 2).

By using Proposition 6, one can show that the function u is a solution of the Cauchy problem (AE), (C) if and only if it is a solution of the integral equation

$$u(t,x) = g(t) + \int_{h(t)}^x \psi(\eta)\mathrm{d}\eta + \int_{h^{-1}(x)}^t \int_{h(s)}^x F(u)(s,\eta)\mathrm{d}\eta\mathrm{d}s$$

in the space $C(\mathcal{D};\mathbb{R})$.

[2]In other words, the pair (g,ψ) is h-consistent (see [5, Section 3]).

3. Main results

The following statements show that both Darboux and Cauchy problem for hyperbolic equation (HE) can be rewritten as initial value problems for abstract equation (AE) in the Banach space $C([c,d];\mathbb{R})$. Consequently, hyperbolic equation (HE) can be regarded as a particular case of abstract equation (AE) with $\mathbb{X} = C([c,d];\mathbb{R})$.

Darboux problem

Theorem 7. *If u is a solution of the problem*

$$\frac{\partial^2 u(t,x)}{\partial t\,\partial x} = F(u)(t,x), \tag{HE}$$

$$u(t,c) = \alpha(t) \quad \text{for } t \in [a,b], \quad u(a,x) = \beta(x) \quad \text{for } x \in [c,d], \tag{D}$$

then the function v defined by the formula $v(t)(x) := u(t,x)$ for $t \in [a,b]$, $x \in [c,d]$ is a solution of the problem

$$v'(t) = G(v)(t), \tag{AE}$$

$$v(a) = \beta \tag{I}$$

in the Banach space $C([c,d];\mathbb{R})$, where

$$\left.\begin{aligned} &G(w)(t) := \widetilde{w}(t) \quad \text{for a. e. } t \in [a,b] \text{ and all } w \in C([a,b];C([c,d];\mathbb{R})),\\ &\widetilde{w}(t)(x) := \alpha'(t) + \int_c^x F(z)(t,\eta)\,\mathrm{d}\eta \quad \text{for a. e. } t \in [a,b] \text{ and all } x \in [c,d],\\ &z(t,x) := w(t)(x) \quad \text{for } (t,x) \in \mathcal{D}. \end{aligned}\right\} \tag{1}$$

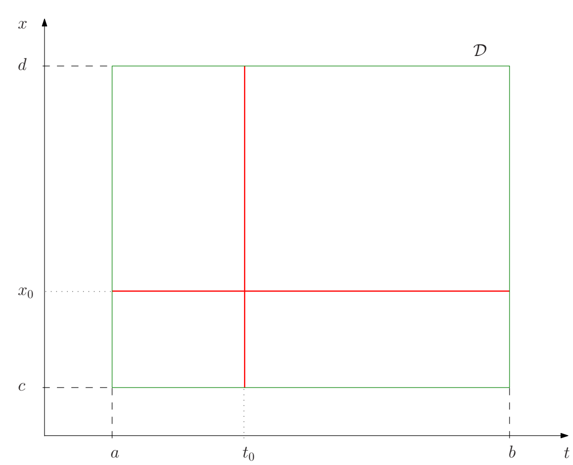
Conversely, if v is a solution of problem (AE), (I) *with G given by* (1), *then the function u defined by the formula $u(t,x) := v(t)(x)$ for $(t,x) \in \mathcal{D}$ is a solution of problem* (HE), (D).

Remark 8. Theorem 7 can be easily extended for "more general" Darboux problem for equation (HE), where the values of the solution u are prescribed on both characteristics $t = t_0$ and $x = x_0$, i. e., the initial conditions are

$$u(t,x_0) = \alpha(t) \quad \text{for } t \in [a,b],$$

$$u(t_0,x) = \beta(x) \quad \text{for } x \in [c,d],$$

where $t_0 \in [a,b]$, $x_0 \in [c,d]$, $\alpha \in AC([a,b];\mathbb{R})$, $\beta \in AC([c,d];\mathbb{R})$ are such that $\alpha(t_0) = \beta(x_0)$.



Cauchy problem

Theorem 9. *If u is a solution of the problem*

$$\frac{\partial^2 u(t,x)}{\partial t\,\partial x} = F(u)(t,x), \tag{HE}$$

$$u(t,h(t)) = g(t) \quad \text{for } t\in[a,b], \quad u'_{[2]}(h^{-1}(x),x) = \psi(x) \quad \text{for a. e. } x\in[c,d], \tag{C}$$

then the function v defined by the formula $v(t)(x) := u(t,x)$ for $t\in[a,b]$, $x\in[c,d]$ is a solution of the problem

$$v'(t) = G(v)(t), \tag{AE}$$

$$v(t)(h(t)) = g(t) \quad \text{for } t\in[a,b] \tag{NL}$$

in the Banach space $C([c,d];\mathbb{R})$, where

$$\left.\begin{aligned} &G(w)(t) := \widetilde{w}(t) \quad \text{for a. e. } t\in[a,b] \text{ and all } w\in C([a,b];C([c,d];\mathbb{R})),\\ &\widetilde{w}(t)(x) := \frac{\mathrm{d}}{\mathrm{d}t}\left(g(t)+\int_{h(t)}^{d}\psi(\eta)\mathrm{d}\eta\right)\\ &\qquad\qquad + \int_{h(t)}^{x} F(z)(t,\eta)\mathrm{d}\eta \quad \text{for a. e. } t\in[a,b] \text{ and all } x\in[c,d],\\ &z(t,x) := w(t)(x) \quad \text{for } (t,x)\in\mathcal{D}. \end{aligned}\right\} \tag{2}$$

Conversely, if v is a solution of problem (AE), (NL) *with G given by* (2)*, then the function u defined by the formula $u(t,x) := v(t)(x)$ for $(t,x)\in\mathcal{D}$ is a solution of problem* (HE), (C).

How to prove Theorems 7 and 9?

Recall that in this part we have $\mathbb{X} = C([c,d];\mathbb{R})$. In order to prove Theorems 7 and 9, we need to discuss and prove in detail the following assertions:

(A) Properties of the relationship between abstract functions and functions of two variables given by the formula

$$\boxed{v(t)(x) = u(t,x) \quad \text{for } t\in[a,b],\ x\in[c,d].}$$

- $u\in C(\mathcal{D};\mathbb{R}) \quad\Rightarrow\quad v\in C([a,b];\mathbb{X})$
- $v\in C([a,b];\mathbb{X}) \quad\Rightarrow\quad u\in C(\mathcal{D};\mathbb{R})$
- $u\in AC(\mathcal{D};\mathbb{R}) \quad\Rightarrow\quad v\in AC([a,b];\mathbb{X})$

  It follows from the definition of the strong absolute continuity of abstract functions and Proposition 6.

- $v \in AC([a,b];\mathbb{X}) \quad \not\Rightarrow \quad u \in AC(\mathcal{D};\mathbb{R})$

  Indeed, let $f \in C([c,d];\mathbb{R})$ be such that $f \notin AC([c,d];\mathbb{R})$ and put
  $$v(t) := f \quad \text{for } t \in [a,b].$$
  Then $v \in AC([a,b];\mathbb{X})$ because it is a constant abstract function. However, $u(t,x) = f(x)$ for $(t,x) \in \mathcal{D}$ which yields that $u(t,\cdot) \notin AC([c,d];\mathbb{R})$ for every $t \in [a,b]$ and thus, $u \notin AC(\mathcal{D};\mathbb{R})$.

- $u \in AC(\mathcal{D};\mathbb{R}) \quad \Rightarrow \quad v'(t) = u'_{[1]}(t,\cdot)$ for a. e. $t \in [a,b]$

  It follows from Proposition 6 that for a. e. $t_0 \in [a,b]$, we have
  $$\lim_{h\to 0} \left| \frac{u(t_0+h,x) - u(t_0,x)}{h} - u'_{[1]}(t_0,x) \right| = 0 \quad \text{for every } x \in [c,d].$$
  The main difficulty here is to prove that for a. e. $t_0 \in [a,b]$, the relation
  $$\lim_{h\to 0} \left| \frac{u(t_0+h,x) - u(t_0,x)}{h} - u'_{[1]}(t_0,x) \right| = 0 \quad \text{uniformly on } [c,d]$$
  holds.

(B) Bochner integrability of the abstract function

$$\boxed{g(t)(x) := p(t) + \int_c^x q(t,\eta)\mathrm{d}\eta \quad \text{for a. e. } t \in [a,b] \text{ and all } x \in [c,d],}$$

where $p \in L([a,b];\mathbb{R})$ **and** $q \in L(\mathcal{D};\mathbb{R})$**.**

- $g(t) \in C([c,d];\mathbb{R})$ for a. e. $t \in [a,b]$ and thus, the abstract function $g\colon [a,b] \to \mathbb{X}$ is defined a. e. on $[a,b]$
- $g \in B([a,b];\mathbb{X})$
- For any $t \in [a,b]$ we have
  $$\int_a^t g(s)\mathrm{d}s \in \mathbb{X}$$
  and, moreover,
  $$\left(\int_a^t g(s)\mathrm{d}s \right)(x) = \int_a^t p(s)\mathrm{d}s + \int_a^t \int_c^x q(t,\eta)\mathrm{d}\eta\mathrm{d}s \quad \text{for } x \in [c,d].$$
  Note that the integrals on the right-hand side are Lebesgue ones.

(C) Properties of the operator G in Theorem 7 and 9

- It follows from the above assertions that the operator G defined by formula (1) (respectively, (2)) maps $C([a,b];\mathbb{X})$ into $B([a,b];\mathbb{X})$, it is continuous and satisfies the local Carathéodory condition as required.

4. One application of Theorem 7

Consider the linear problem

$$\boxed{v'(t) = T(v)(t) + g(t); \quad v(a) = v_0} \tag{3}$$

in the Banach space $\langle \mathbb{X}, \|\cdot\|_{\mathbb{X}} \rangle$, where $T\colon C([a,b];\mathbb{X}) \to B([a,b];\mathbb{X})$ is a linear bounded operator, $g \in B([a,b];\mathbb{X})$, and $v_0 \in \mathbb{X}$. In the sequel we assume that

- $\mathbb{X}$ is endowed with the preordering $\leqslant_K$ generated by a wedge[3] $K \subset \mathbb{X}$, i. e., we have

$$x_1 \leqslant_K x_2 \iff x_2 - x_1 \in K.$$

It is well-known that theorems on differential inequalities (maximum principles in other terminology) play very important role in the study of solvability of initial and boundary value problems for differential equations as well as in the investigation of asymptotic properties of their solutions. For linear problem (3), a maximum principle can be formulated in the following way.

Definition 10. We say that a *maximum principle holds for problem* (3) if the implication

$$\left.\begin{array}{l} v \in AC([a,b];\mathbb{X}), \\ v \text{ is differentiable a. e. on } [a,b], \\ v'(t) \geqslant_K T(v)(t) \text{ for a. e. } t \in [a,b], \\ v(a) \geqslant_K 0 \end{array}\right\} \quad \Rightarrow \quad v(t) \geqslant_K 0 \text{ for } t \in [a,b]$$

is true.

In the sequel we need to equip the Banach space $C([a,b];\mathbb{X})$ with a "strict type inequality":

- for $f \in C([a,b];\mathbb{X})$ we put

$$f \blacktriangleright 0 \iff \forall g \in C([a,b];\mathbb{X})\ \exists \varepsilon > 0 \quad \text{such that} \quad \varepsilon g(t) \leqslant_K f(t) \quad \text{for } t \in [a,b].$$

Theorem 11. *Let T be a B-positive*[4] *operator and there exist a function $\gamma \in AC([a,b];\mathbb{X})$, which is differentiable a. e. on $[a,b]$ and satisfies*

$$\gamma \blacktriangleright 0,$$
$$\gamma'(t) \geqslant_K T(\gamma)(t) \quad \textit{for a. e. } t \in [a,b].$$

Then the maximum principle holds for problem (3).

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[3] A non-empty closed set $K \subset \mathbb{X}$ is called a wedge if

$$\lambda_1 x_1 + \lambda_2 x_2 \in K \quad \text{for } x_1, x_2 \in \mathbb{X},\ \lambda_1, \lambda_2 \in [0, +\infty[\,.$$

[4] The operator T satisfies

$$v \in C([a,b];\mathbb{X}),\ v(t) \geqslant_K 0 \text{ for } t \in [a,b] \quad \Rightarrow \quad \int_a^t T(v)(s)\mathrm{d}s \geqslant_K 0 \text{ for } t \in [a,b].$$

In both cases $\mathbb{X} = \mathbb{R}$ and $\mathbb{X} = \mathbb{R}^n$, Theorem 11 coincides with our results concerning first-order functional differential equations and their systems. We will show a consequence of Theorem 11 for the Darboux problem for linear hyperbolic equations.

Consider the Darboux problem

$$\boxed{\begin{gathered}\frac{\partial^2 u(t,x)}{\partial t\,\partial x} = \ell(u)(t,x) + q(t,x),\\ u(t,c) = \alpha(t) \quad \text{for } t \in [a,b], \quad u(a,x) = \beta(x) \quad \text{for } x \in [c,d],\end{gathered}} \tag{4}$$

where $\ell\colon C(\mathcal{D};\mathbb{R}) \to L(\mathcal{D};\mathbb{R})$ is a linear bounded operator , $q \in L(\mathcal{D};\mathbb{R})$, and the functions α, β are as in Section 2.2 (we have studied this problem, e. g., in [2,4,6,9–11]).

Definition 12. We say that a *strong maximum principle holds for problem* (4) if the implication

$$\left.\begin{aligned}&u \in AC(\mathcal{D};\mathbb{R}),\\ &u''_{[1,2]}(t,x) \geq \ell(u)(t,x) \text{ for a. e. } (t,x) \in \mathcal{D},\\ &u(t,c) \geq 0 \text{ for } t \in [a,b],\\ &u(a,x) \geq 0 \text{ for } x \in [c,d],\\ &\text{either } u'_{[1]}(t,c) \geq 0 \text{ for a. e. } t \in [a,b]\\ &\qquad \text{or } u'_{[2]}(a,x) \geq 0 \text{ for a. e. } x \in [c,d]\end{aligned}\right\} \quad \Rightarrow \quad u(t,x) \geq 0 \text{ for } (t,x) \in \mathcal{D} \tag{5}$$

is true.

Theorems 7 and 11 yield

Proposition 13. *Let ℓ be a positive*[5] *operator and there exist a function $\omega \in AC(\mathcal{D};\mathbb{R})$ satisfying*

$$\omega(t,x) > 0 \quad \textit{for } (t,x) \in \mathcal{D},$$

$$\omega''_{[1,2]}(t,x) \geq \ell(\omega)(t,x) \quad \textit{for a. e. } (t,x) \in \mathcal{D},$$

$$\textit{either} \quad \omega'_{[1]}(t,c) \geq 0 \quad \textit{for a. e. } t \in [a,b] \quad \textit{or} \quad \omega'_{[2]}(a,x) \geq 0 \quad \textit{for a. e. } x \in [c,d].$$

Then the strong maximum principle holds for problem (4).

Sketch of the proof:

- Let $\mathbb{X} := C([c,d];\mathbb{R})$ and

  $$K := \{y \in \mathbb{X} : y(x) \geq 0 \text{ for } x \in [c,d]\},$$

  which yields that for $f \in C([a,b];\mathbb{X})$ we have

  $$f \blacktriangleright 0 \iff f(t)(x) > 0 \quad \text{for } t \in [a,b],\ x \in [c,d].$$

[5] The operator ℓ satisfies

$$u \in C(\mathcal{D};\mathbb{R}),\ u(t,x) \geq 0 \text{ for } (t,x) \in \mathcal{D} \quad \Rightarrow \quad \ell(u)(t,x) \geq 0 \text{ for a. e. } (t,x) \in \mathcal{D}.$$

- We put
$$\gamma(t)(x) := \omega(t,x) \quad \text{for } t \in [a,b],\ x \in [c,d].$$
By virtue of properties (A) and (B) stated in Section 3, we show that the abstract function γ is of the class $AC([a,b];\mathbb{X})$ and satisfies the inequalities required in Theorem 11 with
$$T(w)(t) := \widetilde{w}(t) \quad \text{for a. e. } t \in [a,b] \text{ and all } w \in C([a,b];C([c,d];\mathbb{R})),$$
where
$$\widetilde{w}(t)(x) := \int_c^x \ell(z)(t,\eta)\mathrm{d}\eta \quad \text{for a. e. } t \in [a,b] \text{ and all } x \in [c,d],$$
$$z(t,x) := w(t)(x) \quad \text{for } (t,x) \in \mathcal{D}.$$

- Now let u be a function satisfying the conditions on the left-hand side of implication (5) and assume that $u'_{[1]}(t,c) \geq 0$ for a. e. $t \in [a,b]$. Then u is a solution of the Darboux problem (HE), (D), where
$$F(z)(t,x) := \ell(z)(t,x) + u''_{[1,2]}(t,x) - \ell(u)(t,x) \quad \text{for a. e. } (t,x) \in \mathcal{D} \text{ an all } z \in C(\mathcal{D};\mathbb{R}),$$
$$\alpha(t) := u(t,c) \quad \text{for } t \in [a,b],$$
$$\beta(x) := u(a,x) \quad \text{for } x \in [c,d].$$
It follows from Theorem 7 that the function
$$v(t)(x) := u(t,x) \quad \text{for } t \in [a,b],\ x \in [c,d]$$
is a solution of the abstract initial value problem (AE), (I), where the operator G is given by (1). By using the assumptions on the function u, we get
$$v'(t) \geqslant_K T(v)(t) \quad \text{for a. e. } t \in [a,b], \quad v(a) \geqslant_K 0$$
and thus, Theorem 11 yields that $v(t) \geqslant_K 0$ for $t \in [a,b]$, i. e.,
$$u(t,x) \geq 0 \quad \text{for } (t,x) \in \mathcal{D}.$$

Remark 14. Proposition 13 is in a compliance with [6, Theorem 3.1], where a maximum principle for hyperbolic equations is proved directly.

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