Asymptotic Solution of the Linearized Korteweg-de Vries Equation

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Brno, Differential equations & applications, 2017.

Grant: RSF-16-11-10282

Initial problem.

Consider the linearized KdV equation

$$\psi_t + (C(x,\, t)\psi)_x + h^2\psi_{xxx} = 0,$$

where h is a small parameter which is the characteristics of the dispersion effects of the media, $x \in \mathbb{R}$. For this equation, we pose the Cauchy problem with localized initial data

$$\psi|_{t=0} = V\left(\frac{x-\xi}{\mu}\right),$$

where another small parameter μ is the characteristics of the localization of the source. Function $V(y)$ is smooth and fast decaing and ξ is the fixed point.

We are interested in constructing the asymptotics of the Cauchy problem while $\mu \to 0$.

Linearization of the KdV equation.

Consider the KdV equation

$$C_t + CC_x + h^2 C_{xxx} = 0.$$

Then let function $u(x,\, t) = C(x,\, t) + \psi(x,\, t)$ with unknown pertrubation ψ. After substituting such function into KdV equation we obtain equation for ψ and linearization of this equation leads to the linearized KdV over background $C(x,\, t)$

$$\psi_t + (C(x,\, t)\psi)_x + h^2 \psi_{xxx} = 0.$$

Two cases will be considered: $C(x,\, t) \equiv C = const$ and variable $C(x,\, t)$.

Constant coefficient. Some history.

The linearized KdV with constant coefficient is well known in the field of waves with dispersion.

Whitham (*Linear and non linear waves*) pointed out that this equation has the solution in the form of Airy function.

Haberman (*Applied Partial Differential Equations with Fourier Series and Boundary Value Problems*) has written this solution.

$$\psi(x,\, t) = \frac{1}{t^{1/3}} Ai \left(\frac{x - \xi - Ct}{t^{1/3}} \right)$$

Karpman (*Non-linear waves in dispersive media*) showed that the Airy function is the Green function for liearized KdV

$$G(x,\, t) = \frac{1}{h^{2/3}\sqrt[3]{3t}} Ai \left(\frac{x - \xi - Ct}{h^{2/3}\sqrt[3]{3t}} \right).$$

Ideas for the asymptotics.

For the case of constant coefficients the Fourier method provides the exact solution.

$$\psi(x,\, t) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} \tilde{V}(p) e^{\frac{i}{\mu}(p[(x-\xi)-Ct]+p^3\lambda^2 t)} dp.$$

But for variable coefficients the Fourier method is not constructive. Therefore we want to provide the method of constructing of the asymptotics for the solution based on the Maslov's canonical opertor.

Maslov and Fedoryuk, *Semiclassical Approximation for Equations of Quantum Mechanics*

Littlejohn, *The Van Vleck Formula, Maslov Theory and Phase Space Geometry*, Journal of Statistical Physics, vol. 68: 1/2, pp. 7-50, 1992

The Maslov's canonical operator in simple cases is similar to WKB-solution. The problem to the WKB-type solutions is the critical points. The canonical operator provides the algorithm for dealing with such points.

The initial function and Maslov's canonical operator

Dobrokhotov, Tirozzi, and Shafarevich, *Representations of rapidly decaying functions by the Maslov canonical operator*, Math. Notes 82 (5-6), 713-717 (2007).

Initial function $V\left(\frac{x-\xi}{\mu}\right)$ can be represented via Maslov's canonical operator.

$$V\left(\frac{x-\xi}{\mu}\right) = \frac{1}{\sqrt{2\pi}}\int e^{\frac{i}{\mu}p(x-\xi)}\tilde{V}(p)dp = \sqrt{\frac{\mu}{i}}K^{\mu}_{\Lambda_0}[\tilde{V}],\ \sqrt{i} = e^{i\pi/4},$$

where $\tilde{V}(p)$ if the Fourier tranform of the $V(x)$. And the initial Lagrangian manifold

$$\Lambda_0 = \{x = \xi,\ p = \alpha;\ \alpha \in \mathbb{R}\}$$

is the vertical line.

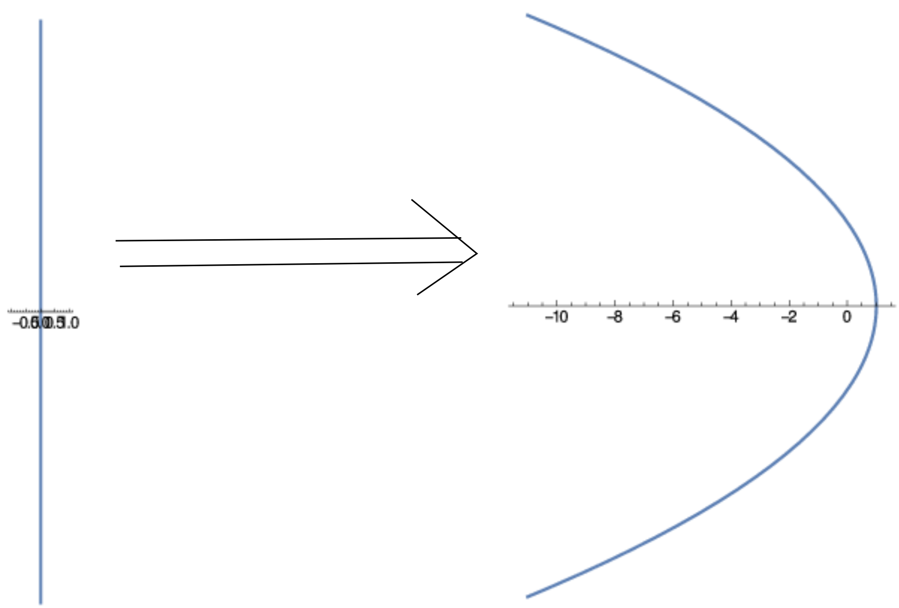
We start working in the phase space $(x,\ p)$ on the Lagrangian manifolds instead of configuration space. Maslov's canonical operator provides mapping from function on manifolds into configuration space.

Hamiltonian system and the Lagrangian manifolds.

The dynamic of the equation in the phase space can be described via the Hamiltonian system.

For linearized KdV with constant coefficient the Hamilton function is of the form

$$H(x, p) = pC - \lambda^2 p^3, \; \lambda = h/\mu.$$

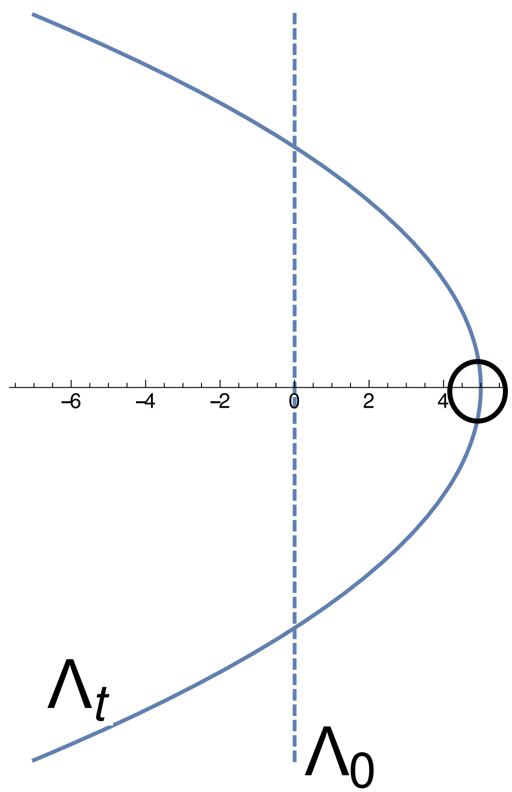


$$\Lambda_t = \{X(\alpha, t), \; P(\alpha, t)\} \equiv \{(C - 3\lambda^2\alpha^2)t + \xi, \; \alpha\}.$$

where X and P is the solution to the Hamiltonian system and $\alpha \in \mathbb{R}$.

Focal charts and the ways of integration.

During the time the focal points ($X_\alpha(\alpha, t) = 0$, the critical points for WKB) may appear at the Lagrangian manifold.



We can devide the manifold for the regular charts (no focal points) and non-regular charts (contains the forcal points).

There is no one-to-one projection of the non-regular charts on the x-plane, but exists projection on the p-plane.

In regular charts Maslov's canonical operator is of the WKB-type. For non-regular charts it has a little bit more complicated form.

The Maslov's canonical operator.

In the regular chart the precanonical operator is the WKB function

$$K^{\mu}_{\Omega_j}[f(\alpha)](x,\, t) = \frac{e^{\frac{i}{\mu}S(\alpha,t)}}{\sqrt{|X_\alpha(\alpha,\, t)|}} f(\alpha)|_{\alpha=\alpha(x,t)},$$

where $f(\alpha)$ is the function on Λ_t, and $\alpha = \alpha(x,\, t)$ is the solution of $x = X(\alpha,\, t)$.

In the non-regular chart the precanonical operator is of the form of the Fourier transform

$$K^{\mu}_{\Omega_j}[f(\alpha)](x,\, t) = \frac{e^{i\pi/4}}{\sqrt{2\pi\mu}} \int \frac{e^{\frac{i}{\mu}(S(\alpha,t)-pX(\alpha,t))}}{\sqrt{|P_\alpha(\alpha,\, t)|}} e^{\frac{i}{\mu}px} f(\alpha)|_{\alpha=\alpha(p,t)} dp,$$

where $\alpha = \alpha(p,\, t)$ is the solution of $p = P(\alpha,\, t)$.

The canonical operator has the form

$$K^{\mu}_{\Lambda_t}[f(\alpha)](x,\, t) = \sum_j e^{-i\frac{\pi}{2}m(\Omega_j)} K^{\mu}_{\Omega_j}[f(\alpha)\, e_j(\alpha)](x,\, t).$$

The asymptotic of the solution.

The asymptotic solution of the Cauchy problem is given by the following formula

$$\psi(x,\,t) = e^{-im(\alpha_0,t)\pi/2}e^{-i\pi/4}\sqrt{\mu}\times$$
$$\times e^{i/\mu\int_0^t\Big(P(\alpha_0,\tau)H_p(X(\alpha_0,\tau),P(\alpha_0,\tau))-H(X(\alpha_0,\tau),P(\alpha_0,\tau))\Big)d\tau}K^{\mu}_{\Lambda_t}[\tilde{V}(\alpha)].$$

Here the $m(\alpha_0,\,t)$ is the Maslov index of the path ($= \pm$ number of focal points or the Morse index).

The general problem is to evaluate and simplify the Maslov's canonical operator.

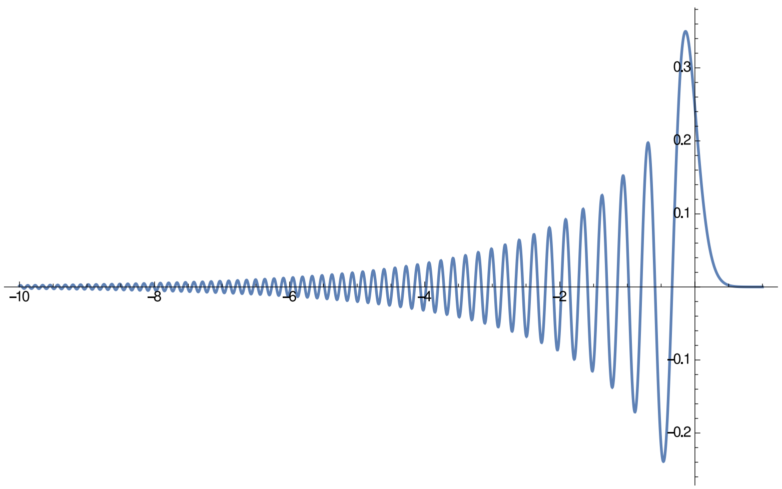
For constant coefficients it is the Fourier transform

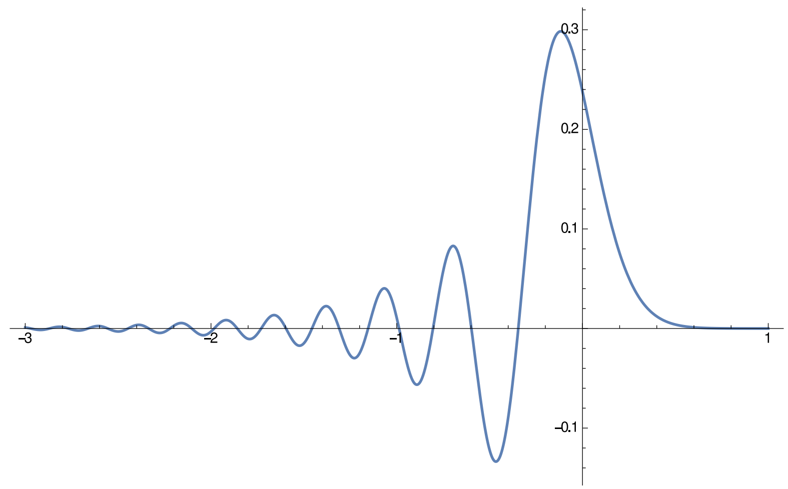
$$\psi(x,\,t) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}\tilde{V}(p)e^{\frac{i}{\mu}(p[(x-\xi)-Ct]+p^3\lambda^2 t)}dp.$$

Solution of the Cauchy problem

If the initial function $V(y)$ is of the form of Gaussian exponential then the exact solution has the form

$$\psi(x,\, t) = \frac{\mu}{h^{2/3}\sqrt[3]{3t}} exp\left(\frac{\mu^6}{108t^2h^4} + \left(\frac{\mu}{h}\right)^2 \frac{x-\xi-Ct}{6t}\right)\times$$
$$\times Ai\left(\frac{1}{h^{2/3}\sqrt[3]{3t}}\left(\frac{\mu^4}{12th^2} + (x-\xi-Ct)\right)\right).$$





Uniformly by μ, h and t.

Asymptotics of the solution.

Dobrokhotov, Makrakis, and Nazaikinskii, *Maslov's canonical operator, Hörmander's formula, and localization of the Berry-Balazs solution in the theory of wave beams*, Theoret. and Math. Phys. 180 (2), 894-916 (2014).

Hörmander, *Fourier Integral Operators I*, Acta Math., vol. 127:79, pp. 79–183, 1971

For an arbitrary function $V(y)$ with smooth Fourier transform while $t \geq t_0 > 0$ the asymptotics has the form

$$\psi(x,\,t) \sim \sqrt{2\pi} g_1 \left(-\frac{\mu^2 \eta}{3th^2} \right) \frac{\mu}{h^{2/3}\sqrt[3]{3t}} Ai \left(\frac{\eta}{h^{2/3}\sqrt[3]{3t}} \right) -$$

$$-i\sqrt{2\pi} g_2 \left(-\frac{\mu^2 \eta}{3th^2} \right) \frac{\mu^{7/3}}{(h^{2/3}\sqrt[3]{3t})^2} Ai' \left(\frac{\eta}{h^{2/3}\sqrt[3]{3t}} \right).$$

$$\eta = x - \xi - Ct.$$

Here

$$g_1(z) = \begin{cases} g_1^+(z), \ z \geq 0 \\ g_1^-(z), \ z \leq 0. \end{cases} \quad , \quad g_2(z) = \begin{cases} g_2^+(z), \ z \geq 0 \\ g_2^-(z), \ z \leq 0, \end{cases}$$

$$g_1^+(z) = \frac{1}{2}\Big(\tilde{V}(\sqrt{z}) + \tilde{V}(-\sqrt{z})\Big), \quad g_2^+(z) = \frac{1}{2\sqrt{z}}\Big(\tilde{V}(\sqrt{z}) - \tilde{V}(-\sqrt{z})\Big),$$

$$g_1^-(z) = 3g_1^+(0) - 3g_1^+(-z) + g_1^+(-2z),$$
$$g_2^-(z) = 3g_2^+(0) - 3g_2^+(-z) + g_2^+(-2z).$$

Example. Let

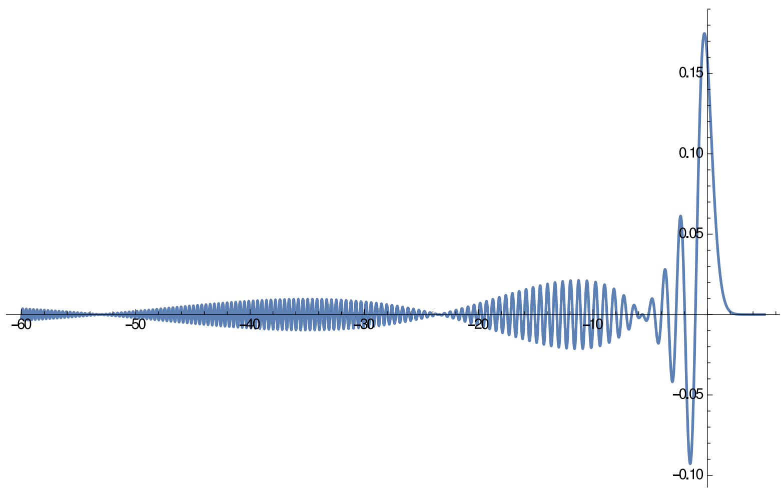
$$V(y) = \frac{1+y}{\cosh y}.$$

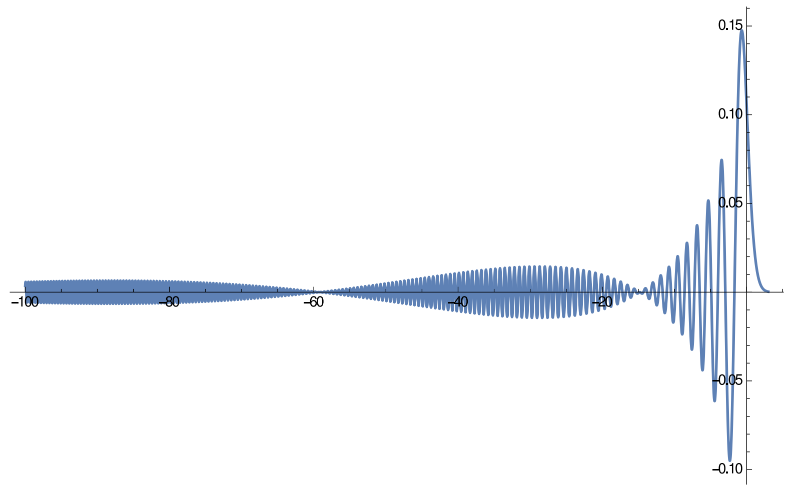
Then when $x < 0$ and $\xi = 0$, $C = 0$

$$\psi(x,\, t) = \frac{\pi h^{1/3}}{\sqrt[3]{3t}} \frac{1}{\cosh(\frac{\pi}{2}\sqrt{-\frac{x}{3t}})} Ai\left(\frac{x}{h^{2/3}\sqrt[3]{3t}}\right) -$$
$$- \frac{2\pi^2 h^{2/3}}{(3t)^{2/3}} \sqrt{-\frac{3t}{x}} \frac{\sinh^3(\frac{\pi}{2}\sqrt{-\frac{x}{3t}})}{\sinh^2(\pi\sqrt{-\frac{x}{3t}})} Ai'\left(\frac{x}{h^{2/3}\sqrt[3]{3t}}\right).$$

Example of the asymptotics.

Initial function $V(y) = 1$, $|y| \leq 1$ and $V(y) = 0$, $|y| > 1$ has the smooth Fourier transform therefore the asymptotics for such Cauchy problem can be evaluated.





The asymptotic consists of the two parts. Near the focal point it has the form of the Airy function and outside the vicinity of such point it has the WKB-type corresponding to the $\sin p/p$.

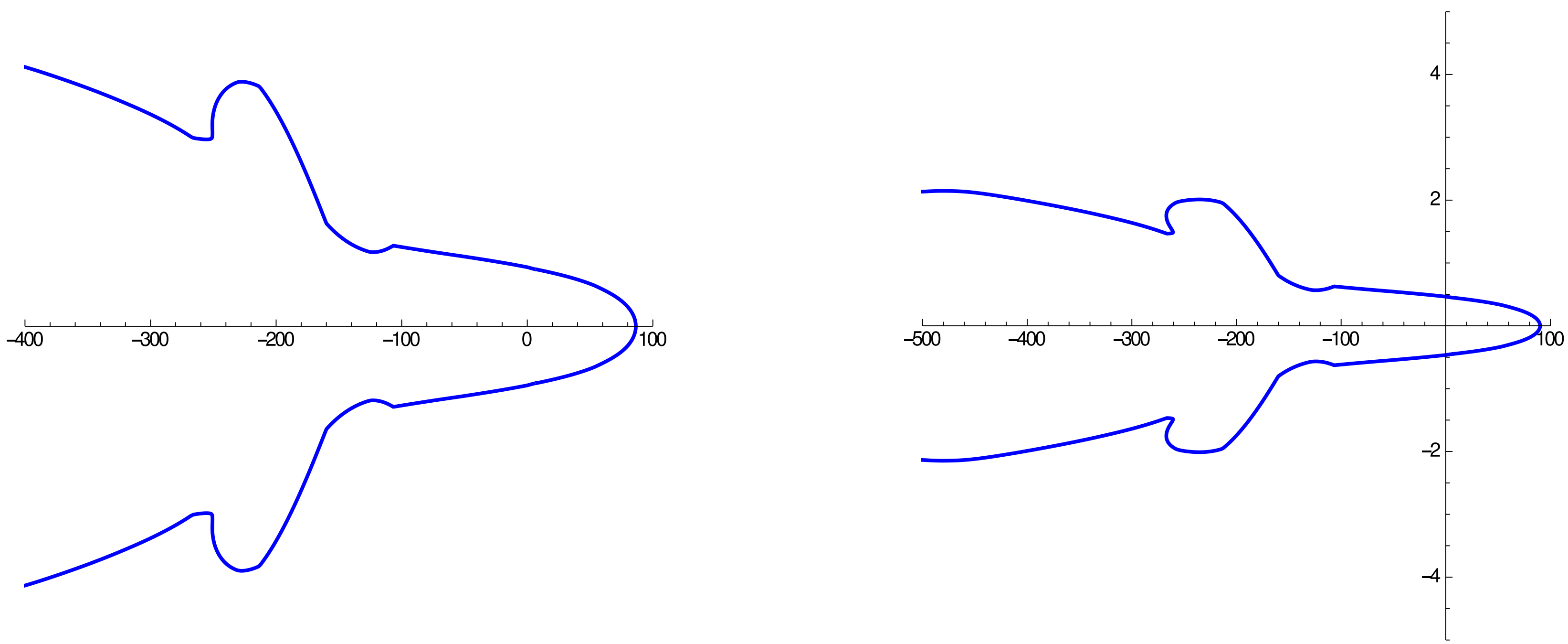
The variable coefficients.

The Hamilton function

$$H(x,\, p,\, t) = C(x,\, t)p - \lambda^2 p^3.$$

Because h is small we choose the background $C(x,\, t)$ as a solution of the Hopf equation instead of KdV.

The Lagrangian manifolds at different times with focal points cusp ($X_{\alpha\alpha} = 0,\, X_{\alpha\alpha\alpha} \neq 0$) and fold ($X_{\alpha\alpha} \neq 0$)



To the cusp corresponds Pearcey function and to the fold — Airy function.

The leading front and waves.

Dobrokhotov and Nazaikinskii, *Punctured Lagrangian manifolds and asymptotic solutions of linear water-wave equations with localized initial conditions*, Math. Notes 101 (5-6), 1053-1060 (2017)

Due to the form of the Hamilton function the fastest velocity is in the neighbourhood of $p = 0$. Near this point the solution of the Hamilton system can be reduced to the following

$$\tilde{X}(\alpha, t) = X_0(t) + \alpha^2 X_2(t),\ \tilde{P}(\alpha, t) = \alpha P_0(t).$$

where for $X_0(t)$ and $P_0(t)$ the Hamiltonian system with reduced Hamilton function $H^0(x, p, t) = C(x, t)p$ holds

$$\dot{x}_0 = C(x_0, t),\ \dot{p}_0 = -C_x(x_0, t)p_0,\ x_0|_{t=0} = \xi,\ p_0|_{t=0} = 1,$$

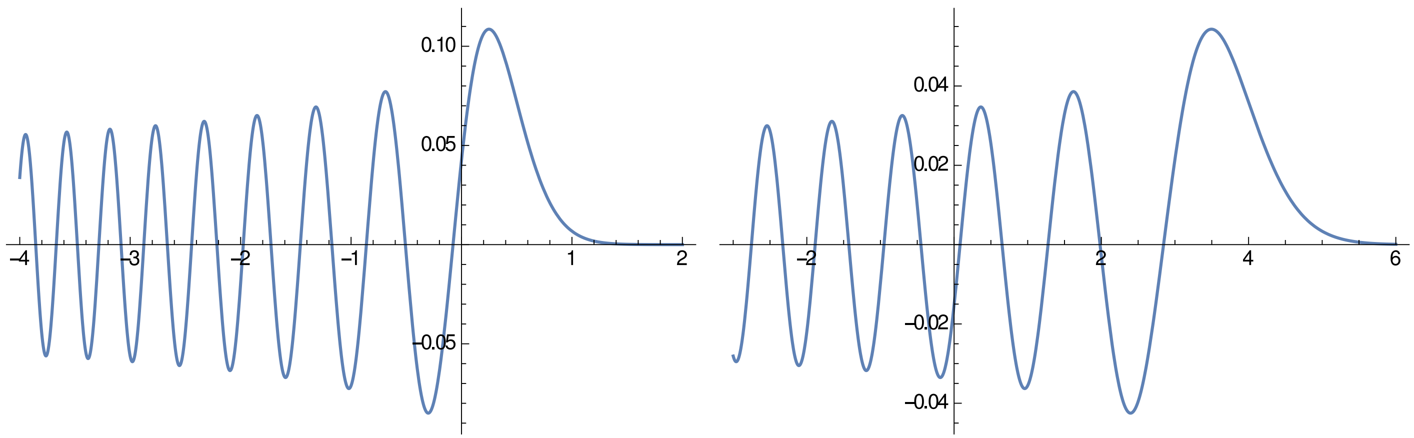
and $X_2(t)$ is the solution of the following equation

$$\dot{x}_2 = C_x(X_0(t), t)x_2 - 3\lambda^2 P_0^2(t),\ x_2|_{t=0} = 0,$$

Asymptotic of the leading wave

The asymptotic of the leading wave for $t \geq t_0 > 0$ is of the form

$$\psi(x,\, t) \sim -\mu\sqrt{2\pi}\tilde{V}(0)\frac{P_0^{2/3}(t)}{\mu^{2/3}\sqrt[3]{X_2(t)}}Ai\left(-P_0^{2/3}\frac{x - X_0(t)}{\mu^{2/3}\sqrt[3]{X_2(t)}}\right)$$



If we choose the constant coefficient then this formula transforms to

$$\psi \sim \mu\sqrt{2\pi}\tilde{V}(0)\frac{1}{h^{2/3}\sqrt[3]{3t}}Ai\left(\frac{x - \xi - Ct}{h^{2/3}\sqrt[3]{3t}}\right).$$

This is the Green function and is particular case of the formula for the constant coefficient.

THANK YOU FOR YOUR ATTENTION !

More details can be found here

S.A. Sergeev, Mathematical Notes, 2017, Vol. 102, No. 3, pp. 111-124