State-dependent impulses and distributions in BVPs

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Motivation

Real world problems where the evolution of systems is affected by rapid changes are modelled by means of differential equations with impulses. Abrupt changes of solutions of impulsive problems imply that such solutions do not preserve the basic properties which are associated with non-impulsive problems. We focus our attention to problems with a finite number $m \in \mathbb{N}$ impulses on the compact interval $[0, T] \subset \mathbb{R}$. Most papers deal with *fixed-time* impulses where the moments of impulses

$$0 < t_1 < t_2 < \cdots < t_m < T$$

are fixed and known before. This is a special case of so called *state-dependent* impulses where the impulse moments and impulse values depend on a solution of a differential equations and different solutions can have different times and sizes of jumps.

We can characterize a dependence of impulse moments $t_1, \dots, t_m$ on solutions as follows:

- Let $\tau_1, \dots, \tau_m$ be functionals defined on a suitable functional space X and having values in $(0, T)$. Then the impulse moments $t_1, \dots, t_m$ are given as

$$t_i = \tau_i(x) \in (0, T), \quad x \in X, \quad i = 1, \dots, m.$$

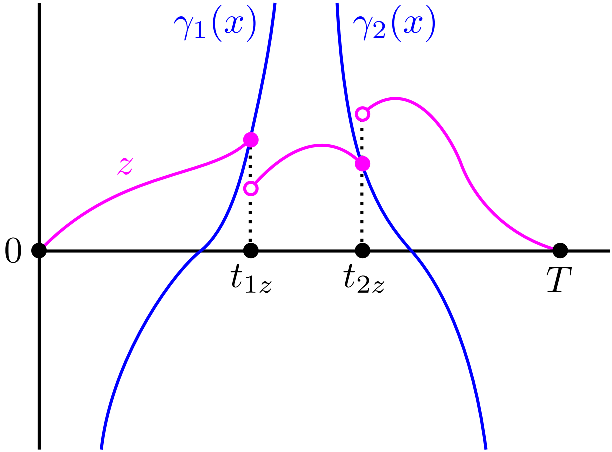
- The functionals $\tau_1, \dots, \tau_m$ can be defined by barriers: Let $\gamma_1, \dots, \gamma_m$ be functions (barriers) defined on a suitable interval $[a, b] \subset \mathbb{R}$ and having values in $(0, T)$. Then the impulse moments $t_1, \dots, t_m$ are given as

$$t_i = \gamma_i(x(t_i)) \in (0, T), \quad x \in X, \quad i = 1, \dots, m.$$

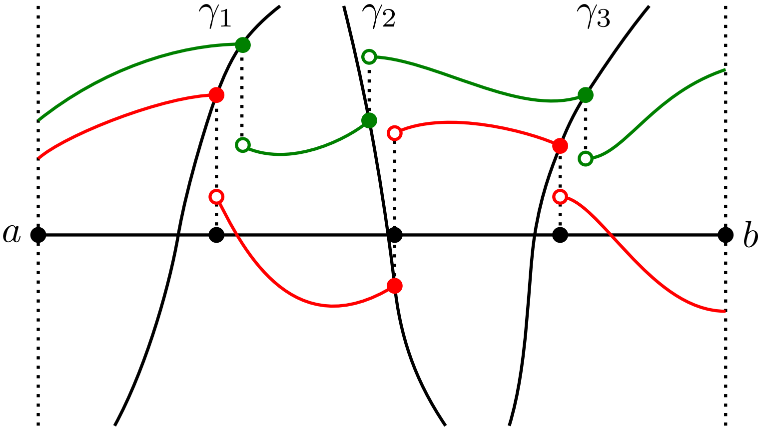
In order to get the desired number m of impulse points in this case it is necessary to impose additional conditions (transversality conditions) on $\gamma_1, \dots, \gamma_m$.

Functionals τ_1, τ_2 via barriers γ_1, γ_2

$$\tau_1(z) = t_{1z}, \quad \tau_2(z) = t_{2z}, \quad z \in \mathrm{BV}.$$



Functionals τ_1, τ_2, τ_3 via barriers γ_1, γ_2, γ_3



Periodic problems

A lot of papers studying impulsive periodic problems are population or epidemic models.

- Differential equations in these models have mostly a form of *autonomous* planar differential systems.
- On the other hand, non-autonomous population or epidemic models are investigated as well, but with fixed-time impulses only.
- However, there are a few existence results for non-autonomous periodic problems with state-dependent impulses.

The first attempts can be seen in the monographs **[SP]** and **[A]** investigating periodic solutions of quasilinear systems with state-dependent impulses.

[SP] A.M. Samoilenko, M.A. Perestyuk, Impulsive Differential Equations. *World Scientific*, Singapore 1995.

[A] M. Akhmet, Principles of Discontinuous Dynamical Systems.*Springer*, New York 2010.

One of the first results that are trackable via Scopus is obtained in **[BL]**, where a scalar nonlinear first order differential equation is studied under the assumptions:

- the existence and uniqueness of a solution of the corresponding initial value problem with state-dependent impulses,
- the existence of lower and upper solutions of the periodic problem with state-dependent impulses.

The method of proof is based on a fixed point theorem for a Poincaré operator.

[BL] I. Bajo, E. Liz, Periodic boundary value problem for first order differential equations with impulses at variable times, *J. Math. Anal. Appl.* 204 (1996), 65–73.

A generalization to a system is done in **[FO]** under the assumption that there exists a solution tube to the problem. The authors applied a fixed point theorem to a multivalued Poincaré operator.

[FO] M. Frigon, D. O'Regan, First order impulsive initial and periodic problems with variable moments, *J. Math. Anal. Appl.* 233 (1999), 730–739.

Further interesting result is reached in **[DDL]**. The authors transformed a linear system with delay and state-dependent impulses to a system with fixed-time impulses and then they proved the existence of positive periodic solutions.

[DDL] A. Domoshnitsky, M. Drakhlin, E. Litsyn, Nonoscillation and positivity of solutions to first-order state-dependent differential equations with impulses in variable moments, *J. Differential Equations* 228 (2006), 39–48.

Recently, Tomeček in **[T]** proved the existence of a periodic solution to a nonlinear second order differential equation with ϕ-Laplacian and state-dependent impulses via lower and upper solutions method.

[T] J. Tomeček, Periodic solution of differential equation with ϕ-Laplacian and state-dependent impulses, *J. Math. Anal. Appl.* 450 (2017), 1029–1046.

All the above problems have a *"classical"* formulation in which impulse conditions are given out of a differential equation.

Let us demonstrate it on the next example.

- Consider $T > 0$ and let $m \in \mathbb{N}$ and τ_i, $\mathcal{J}_i$, $i = 1, \ldots, m$, be functionals defined on the set of T-periodic functions of bounded variation. (For simplicity we do not discuss barriers.)

- Assume that $f(t, x)$ is T-periodic in t and satisfies the Carathéodory conditions on $[0, T] \times \mathbb{R}$.

For the second order differential equation (in the system form) the classical formulation of the T-periodic problem with state-dependent impulses at the points $\tau_i(x) \in (0, T)$ writes as

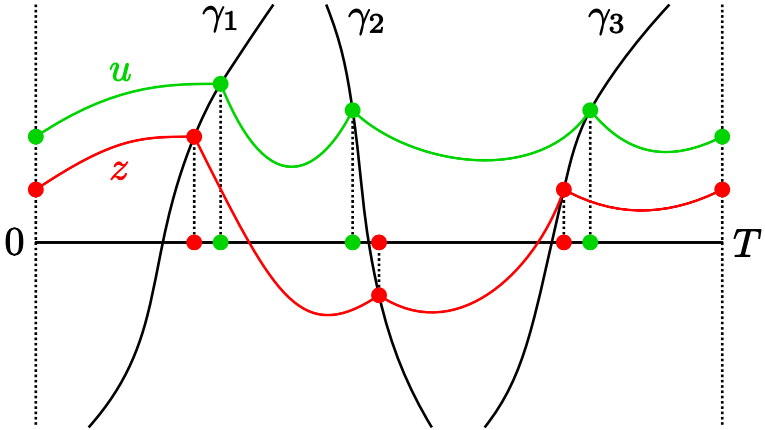
$$x'(t) = y(t), \quad y'(t) = f(t, x(t)) \quad \text{for a.e. } t \in [0, T], \tag{1}$$

$$\triangle y(\tau_i(x)) = \mathcal{J}_i(x), \quad i = 1, \ldots, m, \tag{2}$$

$$x(0) = x(T), \quad y(0) = y(T), \tag{3}$$

where x' and y' denote the classical derivatives of the functions x and y, respectively, $\triangle y(t) = y(t+) - y(t-)$.

Functionals τ_1, τ_2, τ_3 are given by γ_1, γ_2, γ_3 and derivatives of solutions have jumps



Another possible formulation of the T-periodic problem with state-dependent impulses at the points $\tau_i(x) \in (0, T)$ can be written in the form of the *distributional* differential equation

$$D^2 z - f(\cdot, z) = \frac{1}{T} \sum_{i=1}^{m} \mathcal{J}_i(z)\delta_{\tau_i(z)}, \tag{4}$$

where $D^2 z$ denotes the second distributional derivative of a T-periodic function z of bounded variation and $\delta_{\tau_i(z)}$, $i = 1, \dots, m$, are the Dirac T-periodic distributions which involve impulses at the state-dependent moments $\tau_i(z)$, $i = 1, \dots, m$.

Results on the existence of periodic solutions to distributional equations of the type (4) have been reached in

- **[BV1] J. Belley, M. Virgilio**, Periodic Duffing delay equations with state dependent impulses, *J. Math. Anal. Appl.* 306 (2005), 646–662.
- **[BV2] J. Belley, M. Virgilio**, Periodic Liénard–type delay equations with state-dependent impulses, *Nonlinear Analysis* 64 (2006), 568–589.
- **[BG] J.Belley, R. Guen**, Periodic van der Pol equation with state-dependent impulses, *J. Math. Anal. Appl.* 426 (2015) 995–1011.

- In **[BV1]** and **[BV2]** the authors get the existence and uniqueness of solutions to distributional equations which contain also first derivatives and delay. Their approach essentially depends on the global Lipschitz conditions for data functions in order to get a contractive operator corresponding to the problem.

- In **[BG]** the distributional van der Pol equation with the term $\mu(x - x^3/3)'$ which does not satisfy the global Lipschitz condition is studied. For a sufficiently small value of the parameter μ and $m = 1$, the authors find a ball and a contractive operator on this ball, which yields a unique periodic solution.

In order to present exact definitions of solutions for the both cases (classical and distributional) we use the following notation:

- By C^∞ we denote the vector space of all real-valued T-periodic functions of one real variable having continous derivatives of all orders on $\mathbb{R}$. The elements of C^∞ are called *test functions* and C^∞ is equipped with locally convex topological space structure.
- Its topological dual will be denoted by $\mathcal{D}$. The elements of $\mathcal{D}$ are called *T-periodic distributions* or only *distributions*, i.e. these elements are real-valued continuous linear functionals on C^∞.
- For a distribution $u \in \mathcal{D}$ and a test function $\varphi \in \mathrm{C}^\infty$, the symbol $\langle u, \varphi \rangle$ stands for a value of the distribution u at φ.

- The distributional derivative Du of $u \in \mathcal{D}$ is a distribution which is defined by

$$\langle Du, \varphi \rangle = -\left\langle u, \varphi' \right\rangle \quad \text{for each} \quad \varphi \in \mathrm{C}^\infty.$$

- I is the inverse to D on the set of all distributions with zero mean value.
- The Dirac T-periodic distribution δ is defined by

$$\langle \delta, \varphi \rangle = \varphi(0) \quad \text{for each} \quad \varphi \in \mathrm{C}^\infty.$$

All functional spaces defined below consist of *real-valued T-periodic functions*. Clearly it suffices to prescribe their values on some semiclosed interval with the length equal to T.

- BV is the space of functions of bounded variation; the total variation of $x \in \mathrm{BV}$ is denoted by $\mathrm{var}(x)$; for $x \in \mathrm{BV}$ we also define $\|x\|_\infty := \sup\{|x(t)| : t \in [0, T]\}$,
- NBV is the space of functions from BV normalized in the sense that $x(t) = \frac{1}{2}(x(t+) + x(t-))$,
- $\widetilde{\mathrm{NBV}}$ represents the Banach space of functions from NBV having zero mean value ($\overline{x} := \frac{1}{T}\int_0^T x(t)\,\mathrm{d}t = 0$), which is equipped with the norm equal to the total variation $\mathrm{var}(x)$,
- for an interval $J \subset [0, T]$ we denote by $\mathrm{AC}(J)$ the set of absolutely continuous functions on J, and if $J = [0, T]$ we simply write AC,
- for finite $\Sigma \subset [0, T)$ we denote by $\widetilde{\mathrm{PAC}}_\Sigma$ the set of all functions x having zero mean value and such that $x \in \mathrm{AC}(J)$ for each interval $J \subset [0, T]$ for which $\Sigma \cap J = \emptyset$.

- The condition (2) $\triangle y(\tau_i(x)) = \mathcal{J}_i(x), \quad i = 1, \dots, m,$ is not well-posed if $m > 1$ and there exist $i, j \in \{1, \dots, m\}$, $x \in \mathrm{NBV}$ such that $\mathcal{J}_i(x) \neq \mathcal{J}_j(x)$ and $\tau_i(x) = \tau_j(x)$. This case can be treated by assuming additional conditions on τ_i in the form

$$\tau_i(x) \neq \tau_j(x) \quad \text{for } x \in \mathrm{NBV}, \; i, j = 1, \dots, m, \; i \neq j. \tag{5}$$

- Then, for $x \in \mathrm{NBV}$ we introduce a finite set

$$\Sigma_x = \{\tau_1(x), \tau_2(x), \dots, \tau_m(x)\} \tag{6}$$

and under the assumption (5) we can define a solution to the periodic state-dependent impulsive problem (1)-(3).

Definition 1

Let us assume (5). A vector function $(x, y) \in \mathrm{AC} \times \widetilde{\mathrm{PAC}}_{\Sigma_x}$ is a *solution* of problem (1)-(3), if x and y fulfil (1) for a.e. $t \in [0, T]$ and the state-dependent impulse condition (2) is satisfied.

$$(1) \quad x'(t) = y(t), \quad y'(t) = f(t, x(t)) \quad \text{for a.e. } t \in [0, T],$$

$$(2) \quad \triangle y(\tau_i(x)) = \mathcal{J}_i(x), \quad i = 1, \dots, m.$$

We see that $\mathcal{J}_i(x)$ are size of jumps of y at the moments $\tau_i(x) \in (0, T)$, $i = 1, \dots, m$, and that the periodic condition (3) is fulfilled due to the definition of the spaces AC and $\widetilde{\mathrm{PAC}}_{\Sigma_x}$.

Definition 2

Let us define a solution to the distributive equation

$$D^2 z - f(\cdot, z) = \frac{1}{T}\sum_{i=1}^{m} \mathcal{J}_i(z)\delta_{\tau_i(z)}. \tag{4}$$

A function $z \in \mathrm{NBV}$ is a *solution* of the distributional equation (4), if (4) is satisfied in the sense of distributions, i.e.

$$\langle D^2 z - f(\cdot, z), \varphi \rangle = \frac{1}{T}\sum_{i=1}^{m} \mathcal{J}_i(z)\varphi(\tau_i(z)) \quad \text{for each } \varphi \in \mathrm{C}^\infty.$$

In **[RT1]** we proved an equivalence between distributional differential equations and periodic problems with state-dependent impulses and we used this result in **[RT2]**, where we investigated periodic solutions of the van der Pol equation and generalized the results from **[BG]** onto a non-Lipschitz case.

[RT1] I. Rachůnková, J. Tomeček, Equivalence between distributional differential equations and periodic problems with state-dependent impulses, submitted.

[RT2] I. Rachůnková, J. Tomeček, Distributional van der Pol equation with state-dependent impulses, submitted.

Equivalence of problems

Theorem

Let us assume (5), *i.e.*

$$\tau_i(x) \neq \tau_j(x) \quad \textit{for } x \in \mathrm{NBV},\ i,j = 1,\dots,m,\ i \neq j.$$

- *Let* $z \in \mathrm{NBV}$ *be a solution of the distributional differential equation* (4). *Then there exists unique* $(x, y) \in \mathrm{AC} \times \widetilde{\mathrm{PAC}}_{\Sigma_x}$ *such that* $x = z$, $y = Dz$ *a.e. on* $[0, T]$ *and* (x, y) *is a solution of the periodic problem* (1)-(3) *with state-dependent impulses.*
- *Conversely, let* $(x, y) \in \mathrm{AC} \times \widetilde{\mathrm{PAC}}_{\Sigma_x}$ *be a solution of* (1)-(3). *Then* $z = x$ *is a solution of* (4).

Van der Pol equation with state-dependent impulses

Periodic solutions

$$D^2 z = \mu D\left(z - \frac{z^3}{3}\right) - z + f + \frac{1}{T}\sum_{i=1}^{m} \mathcal{J}_i(z)\delta_{\tau_i(z)}. \qquad (7)$$

Here $T > 0$, $m \in \mathbb{N}$ and τ_i, $\mathcal{J}_i$, $i = 1, \dots, m$, are functionals defined on the set of T-periodic functions with a bounded variation. Symbol $D^2 z$ denotes the second distributional derivative of a T-periodic function z with a bounded variation and $\delta_{\tau_i(z)}$ are the Dirac T-periodic distributions which involve impulses at the state-dependent points $\tau_i(z)$, $i = 1, \dots, m$.

We investigate the existence of solutions of the distributional differential equation (7) under the basic assumptions

$$T > 0, \quad \mu > 0, \quad f \in \mathrm{L}^1 \quad \text{(is } T\text{-periodic)},$$

$$\tau_i : \mathrm{NBV} \to [a, b] \subset (0, T), \quad i = 1, \dots, m, \text{ are continuous}$$

and

$$\mathcal{J}_i : \mathrm{NBV} \to [-a_i, a_i] \subset \mathbb{R}, \quad i = 1, \dots, m, \text{ are continuous.}$$

Definition

A function $z \in \mathrm{NBV}$ is called a *solution* of the distributional differential equation (7) if it satisfies (7) in the sense of distributions, i.e.

$$\langle D^2 z, \varphi \rangle = \left\langle \mu D \left(z - \frac{z^3}{3} \right) - z + f, \varphi \right\rangle + \frac{1}{T} \sum_{i=1}^{m} \mathcal{J}_i(z) \varphi(\tau_i(z))$$

for each $\varphi \in \mathrm{C}^\infty$.

Theorem

Assume that $T \in (0, 2\sqrt{3})$. *Then there exists* $\mu_0 > 0$ *such that for each* $\mu \in (0, \mu_0)$:

- *the distributional differential equation* (7) *has at least one solution* z,
- *if* $\tau_i(z) \neq \tau_j(z)$ *for* $i \neq j$, *then* (x, y), *where* $x = z$ *and* $y = Dz$ *a.e. on* $[0, T]$, *is a solution of the periodic problem with state-dependent impulses*

$$x'(t) = y(t), \quad y'(t) = \mu\left(x(t) - x^3(t)/3\right)' - x(t) + f(t),$$

$$\triangle y(\tau_i(x)) = \mathcal{J}_i(x), \quad i = 1, \ldots, m,$$

$$x(0) = x(T), \quad y(0) = y(T).$$

Specification of μ_0

Denote

$$c_0 := |\bar{f}| + \sum_{i=1}^{m} a_i, \quad c_1 := \|\tilde{f}\|_{\mathrm{L}^1} + \sum_{i=1}^{m} a_i,$$

$$T_0 := 1 - \frac{\mu T}{2} - \frac{T^2}{12}, \quad c_2 := \sqrt{\frac{T_0}{\mu T}} - c_0 .$$

The constant μ_0 is a unique solution of the equation

$$\frac{T^2}{4} c_1 = T_0 \left(\frac{2}{3} \sqrt{\frac{T_0}{\mu T}} - c_0 \right).$$

Anti-periodic problems

The study of anti-periodic solutions is closely related to the study of periodic solutions and their existence plays an important role in characterizing behaviour of nonlinear differential equations.
We refer to

- first order anti-periodic differential systems with fixed-time impulses in

B. Ahmad, J.J. Nieto, Existence and approximation of solutions for a class of nonlinear impulsive functional differential equaitons with anti-periodic boundary conditions, *Nonlinear Anal.*, 69 (2008), 3291–3298.

Z. Luo, J. Shen and J.J. Nieto, Antiperiodic boundary value problem for first-order impulsive ordinary differential equations, *Comput. Math. Appl.* 49 (2005), 253–261.

- neural networks which are modeled by anti-periodic systems with fixed-time impulses for example in

A. Abdurahman and H. Jiang, The existence and stability of the anti-periodic solution for delayed Cohen-Grossberg neural networks with impulsive effects, *Neurocomputing*, 149 (2015), 22–28.

P. Shi and L. Dong, Existence and exponential stability of anti-periodic solutions of Hopfield neural networks with impulses, *Appl. Math. Comput.* 216 (2010), 623–630.

C. Xu, Existence and exponential stability of anti-periodic solutions in cellular neural networks with time-varying delays and impulsive effects, *Electron. J. Differ. Equ.*, 2 (2016), 1–14.

- state-dependent impulses in anti-periodic systems modeling neural nerworks where Lipschitz nonlinearities are assumed in

M. Sayli and E. Yilmaz, Anti-periodic solutions for state-dependent impulsive recurrent neural networks with time-varying and continuously distributed delays, *Ann. Oper. Res.*, (2016), 1–27.

Second order differential equations can serve as physical models, for example: Rayleigh equation (acoustics), Duffing, Liénard or van der Pol equations (oscillation theory).

Let us mention:

- anti-periodic solutions of Rayleigh equation with fixed-time impulses in

Li, Y. and Zhang, T., Existence and uniqueness of anti-periodic solution for a kind of forced Rayleigh equation with state dependent delay and impulses, *Commun. Nonlinear. Sci. Numer. Simulat.* 15 (2010), 4076–4083.

- anti-periodic solutions of these equations without impulses in

Chen, T. and Liu, W. and Zhang, J., The existence of anti-periodic solutions for high order Duffing equation, *J. Appl. Math. Comput.* 27 (2008), 271–280.

Li, Y. and Huang, L., Anti-periodic solutions for a class of Liénard-type systems with continuously distributed delays, *Nonlinear Analysis, RWA* 10 (2009), 2127–2132.

Lv, X. and Yang, P. and Liu, D., Anti-periodic solutions for a class of nonlinear second-order Rayleigh equations with delays, *Commun. Nonlinear. Sci. Numer. Simulat.* 15 (2010), 3593–3598.

Xu, Ch. and Liao, M., Antiperiodic Solutions for a Kind of Nonlinear Duffing Equations with a Deviating Argument and Time-Varying Delay, *Adv. Math. Phys.* (2014), 1–7, Article ID 734632.

The first result about the existence and uniqueness of anti-periodic solutions of the distributional Liénard equation with state-dependent imupulses has been reached in **[BB]** under the assumption that functionals describing moments and values of impulses are globally Lipschitz continuous and bounded.

[BB] J. Belley, E. Bondo, Anti-periodic solutions of Liénard equations with state dependent impulses, *J. Differ. Equations* 261 (2016), 4164-4187.

Motivated by our results for periodic problems we generalize the results of the paper **[BB]** onto a non-Lipschitz case. In particular, we focus our considerations on anti-periodic solutions of the van der Pol equation with state-dependent impulses both in "classical" and distributional formulations and proved existence results which are contained in

I. Rachůnková, J. Tomeček, Antiperiodic solution of van der Pol equation with state-dependent impulses, submitted.

Van der Pol equation with state-dependent impulses

Anti-periodic solutions

$$D^2 z = \mu D\left(z - \frac{z^3}{3}\right) - z + f + \frac{1}{2T}\sum_{i=1}^{m} \mathcal{J}_i(z)\varepsilon_{\tau_i(z)}. \qquad (8)$$

Here $T > 0$, $m \in \mathbb{N}$ and τ_i, $\mathcal{J}_i$, $i = 1, \dots, m$, are functionals defined on the set of $2T$-periodic functions with a bounded variation and

$$\varepsilon_\tau := \delta_\tau - \mathcal{T}_T \delta_\tau, \quad \tau \in \mathbb{R}.$$

Symbol $D^2 z$ denotes the second distributional derivative of a $2T$-periodic function z with a bounded variation and $\delta_{\tau_i(z)}$ are the Dirac $2T$-periodic distributions which involve impulses at the state-dependent points $\tau_i(z)$, $i = 1, \dots, m$.

We investigate the existence of solutions of the distributional differential equation (8) under the basic assumptions

$$T > 0, \quad \mu > 0, \quad f \in \mathrm{L}^1 \text{ is } T\text{-anti-periodic},$$

$$\tau_i : \widetilde{\mathrm{NBV}} \to [a, b] \subset (0, T), \quad i = 1, \dots, m, \text{ are continuous}$$

and

$$\mathcal{J}_i : \widetilde{\mathrm{NBV}} \to [-a_i, a_i] \subset \mathbb{R}, \quad i = 1, \dots, m, \text{ are continuous.}$$

Definition

A function $z \in \widetilde{\mathrm{NBV}}$ is called a *solution* of the distributional differential equation (8) if it satisfies (8) in the sense of distributions, i.e.

$$\langle D^2 z, \varphi \rangle = \left\langle \mu D \left(z - \frac{z^3}{3} \right) - z + f + \frac{1}{2T} \sum_{i=1}^{m} \mathcal{J}_i(z) \varepsilon_{\tau_i(z)}, \varphi \right\rangle$$

for each $\varphi \in \mathrm{C}^\infty$.

Theorem

Assume that $T \in (0, \sqrt{3})$. *Then there exists* $\mu_0 > 0$ *such that for each* $\mu \in (0, \mu_0)$:

- *the distributional differential equation* (8) *has at least one solution* z,
- *if* $\tau_i(z) \neq \tau_j(z)$ *for* $i \neq j$, *then* (x, y), *where* $x = z$ *and* $y = Dz$ *a.e. on* $[0, T]$, *is a solution of the anti-periodic problem with state-dependent impulses*

$$x'(t) = y(t), \quad y'(t) = \mu\left(x(t) - x^3(t)/3\right)' - x(t) + f(t),$$

$$\triangle y(\tau_i(x)) = \mathcal{J}_i(x), \quad i = 1, \ldots, m,$$

$$x(0) = -x(T), \quad y(0) = -y(T).$$

Summary

The above existence results are reached by means of distributional equations which are constructed such that they are equivalent with a studied problem.

- T-periodic van der Pol equation with state-dependent impulses

$$D^2 z = \mu D\left(z - \frac{z^3}{3}\right) - z + f + \frac{1}{T}\sum_{i=1}^{m} \mathcal{J}_i(z)\delta_{\tau_i(z)}.$$

- T-anti-periodic van der Pol equation with state-dependent impulses

$$D^2 z = \mu D\left(z - \frac{z^3}{3}\right) - z + f + \frac{1}{2T}\sum_{i=1}^{m} \mathcal{J}_i(z)\varepsilon_{\tau_i(z)}.$$

- **Periodic problem**

$$x'(t) = y(t), \quad y'(t) = \mu \left(x(t) - x^3(t)/3\right)' - x(t) + f(t),$$

$$\triangle y(\tau_i(x)) = \mathcal{J}_i(x), \quad i = 1, \dots, m,$$

$$x(0) = x(T), \quad y(0) = y(T).$$

τ_i, $\mathcal{J}_i$, $i = 1, \dots, m$, are functionals defined on the set of T-periodic functions with a bounded variation, f is Lebesgue interable and T-periodic.

- **Anti-periodic problem**

$$x'(t) = y(t), \quad y'(t) = \mu \left(x(t) - x^3(t)/3\right)' - x(t) + f(t),$$

$$\triangle y(\tau_i(x)) = \mathcal{J}_i(x), \quad i = 1, \dots, m,$$

$$x(0) = -x(T), \quad y(0) = -y(T).$$

τ_i, $\mathcal{J}_i$, $i = 1, \dots, m$, are functionals defined on the set of $2T$-periodic functions with a bounded variation, f is Lebesgue integrable and T-anti-periodic.

Benefits of the distributional approach

Compared to the classical formulation the distributional approach allows us to use the space NBV and get better apriori estimates and better properties of an operator corresponding to a studied problem.

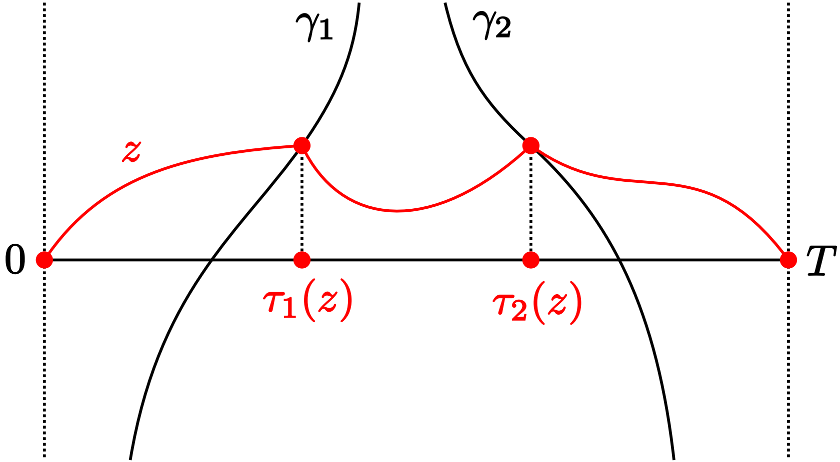
Open problems in the distributional approach

- Consider functionals $\tau_1, \dots, \tau_m$ stated by barriers. Assume that $\gamma_1, \dots, \gamma_m$ are functions (barriers) defined on a suitable interval $[a, b] \subset \mathbb{R}$ and having values in $(0, T)$. Then the impulse moments are given as solutions of

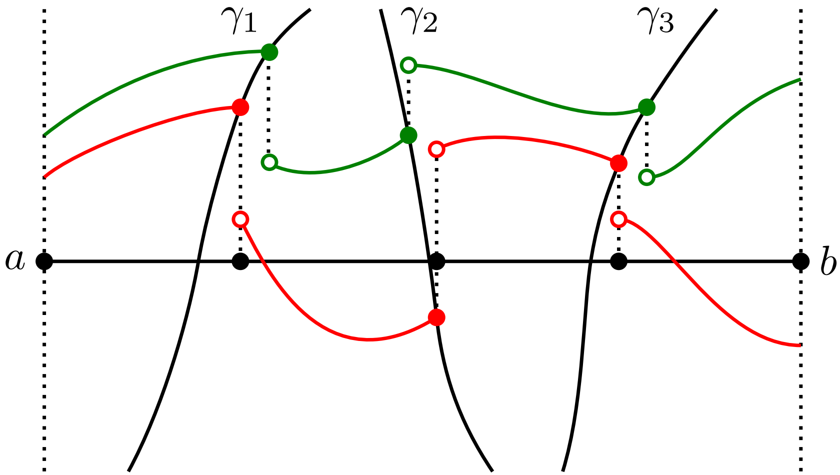
$$t_i = \gamma_i(x(t_i)) \in (0, T), \quad x \in X, \quad i = 1, \dots, m.$$

- Consider impulses not only in Dz but also in z.
- Consider other type of boundary conditions.
- Consider systems of differential equations.

Functionals τ_1, τ_2 stated by barriers γ_1, γ_2, Dirichlet boundary conditions

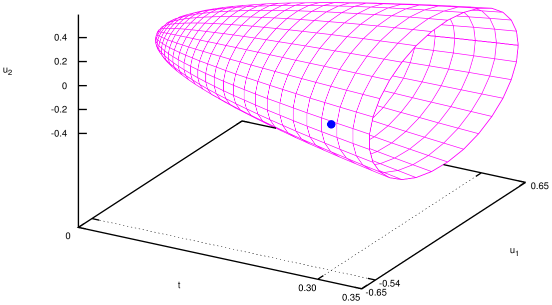


Impulses not only in *Dz* but also in *z*



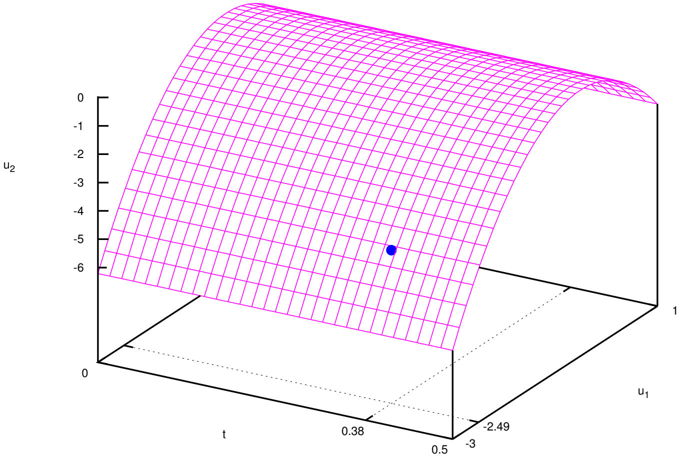
System of two differential equations, 1 barrier $t = \gamma(x_1, x_2)$

$$u_i'(t) = f_i(t, u_1(t), u_2(t)) \text{ for a.e. } t \in (a, b),$$
$$u_i(t+) - u_i(t) = J_i(u_1(t), u_2(t)) \text{ for } t \in (a, b) \text{ such that}$$
$$t = \gamma(u_1(t), u_2(t)), \quad i = 1, 2.$$

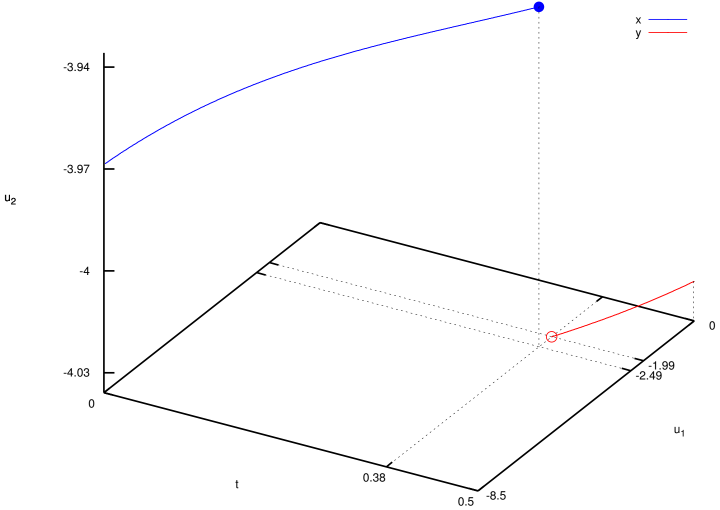


System of two differential equations,

1 barrier: $\gamma(x_1, x_2) = (x_1 + \frac{1}{2})^2 + x_2 - \frac{1}{25} = 0$



Graph of solution $\mathbf{u} = (u_1, u_2)$.



Classical formulation of problems

- **I. Rachůnková, J. Tomeček**, State-Dependent Impulses: Boundary Value Problems on Compact Interval. *Atlantis Press*, Paris 2015. (10 papers)
- **A. Rontó , I. Rachůnková, M. Rontó, L. Rachůnek**, Investigation of solutions of state-dependent multi-impulsive boundary value problem. *Georgian Mathematical Journal*, DOI:10.1515/gmj-2016-0084.
- **I. Rachůnková, L. Rachůnek, A. Rontó, M. Rontó**, A constructive approach to boundary value problems with state-dependent impulses. *Applied Mathematics and Computation* 274 (2016) 726-744.

Thank you for your attention!