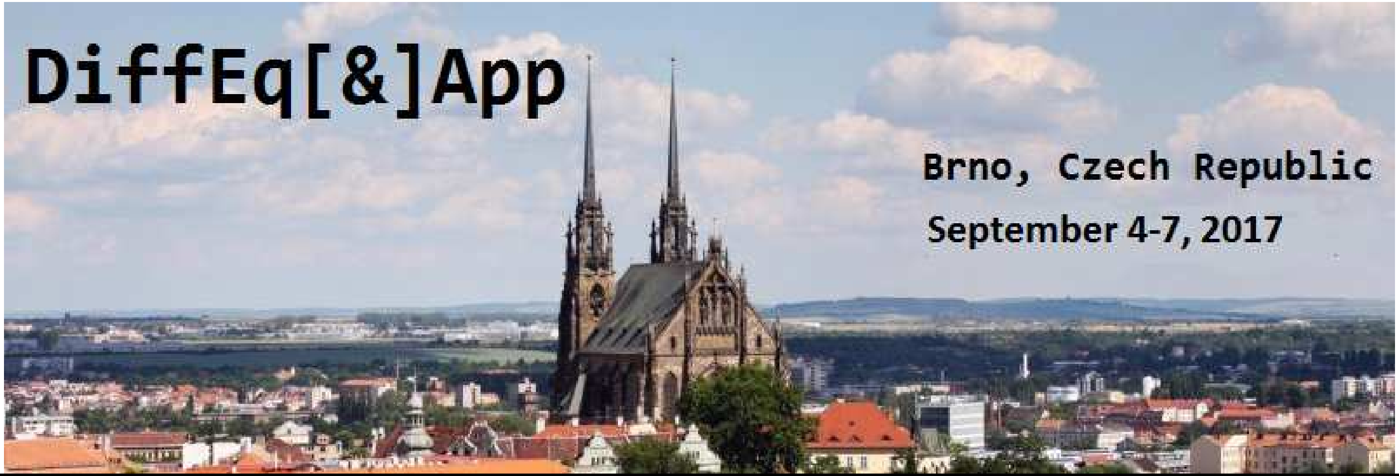

Turing-like phenomenon on a discrete space-time

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Differential Equations and Applications
September 6, 2017

Supported by Interdisciplinary research project MUNI/M/186/2014

Motivations

Generative historiography of ancient Mediterranean

The study of the diffusion dynamics of religious ideas and forms of behavior

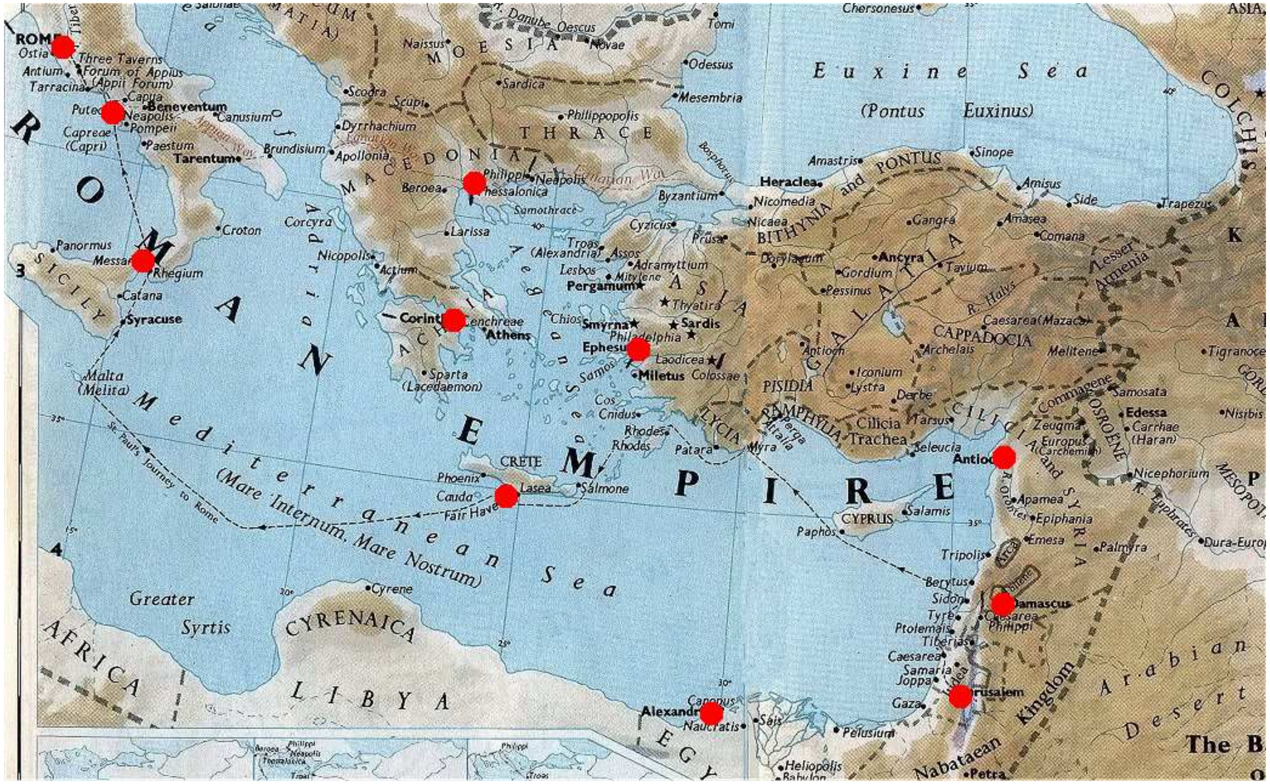
Generative historiography of ancient Mediterranean

The study of the diffusion dynamics of religious ideas and forms of behavior

- E.M.Rogers. *Diffusion of Innovations.* The Free Press, London 1983
- J.M.Epstein *Generative Social Science.* Princeton Univ. Press, 2006
- A.Collar *Religious Networks in the Roman Empire. The spread of New Ideas.* Cambridge Univ. Press, 2013

Generative historiography of ancient Mediterranean

Assumptions:



Assumptions:

- There is a dynamics of interacting ideas in some centers (sites)

Generative historiography of ancient Mediterranean

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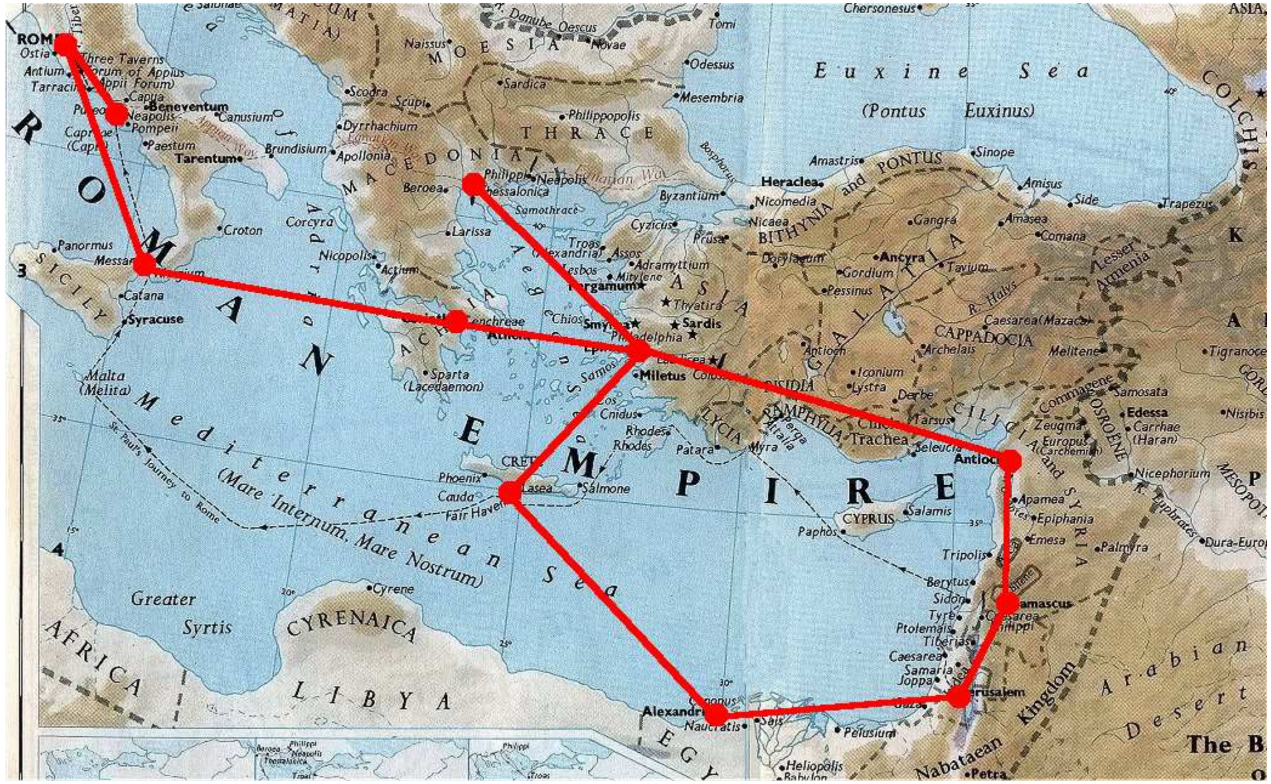
Generative historiography of ancient Mediterranean



Assumptions:

- There is a dynamics of interacting ideas in some centers (sites)
- The ideas spread into neighbor centers
- The ancient Mediterranean is small, the sites are connected

Turing instability

System of reaction-diffusion PDEs

$$\begin{aligned} \frac{\partial u}{\partial t} &= \nabla^2 u + \gamma f(u,v), \\ \frac{\partial v}{\partial t} &= d\nabla^2 v + \gamma g(u,v), \end{aligned} \qquad \boldsymbol{x} \in \Omega, \quad t > 0,$$

$$\frac{\partial u}{\partial \boldsymbol{\nu}} = \frac{\partial v}{\partial \boldsymbol{\nu}} = 0, \qquad \boldsymbol{x} \in \partial\Omega, \quad t > 0.$$

$d > 0$, $\gamma > 0$.

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Steady state of the reaction: $(u^*, v^*) \in \mathbb{R}^2_+$ such that $f(u^*, v^*) = 0 = g(u^*, v^*)$

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$a_{11} := f_u(u^*, v^*)$, $a_{12} := f_v(u^*, v^*)$, $a_{21} := g_u(u^*, v^*)$, $a_{22} := g_v(u^*, v^*)$,

$D := (da_{11} + a_{22})^2 - 4d \det \mathsf{A}$.

Turing instability

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$\det \mathsf{A} > 0$, $a_{11} + a_{22} < 0 \Rightarrow$ the steady state (u^*, v^*) of the ODE system

$$u' = f(u,v), \;\; v' = g(u,v)$$

is stable.

System of reaction-diffusion PDEs

$$
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$\det \mathsf{A} > 0$, $a_{11} + a_{22} < 0$, $da_{11} + a_{22} > \frac{4d}{\gamma} \det \mathsf{A}$,

there exists λ_n, $\nabla^2 w = \lambda_n w, \quad \boldsymbol{x} \in \Omega,$

$$\frac{\partial w}{\partial \nu} = 0 \qquad \boldsymbol{x} \in \partial\Omega,$$

such that $\left| \frac{2d}{\gamma}\lambda_n - da_{11} - a_{12} \right| < \sqrt{D}$

$\Rightarrow$ spatially homogeneous equilibrium of the PDEs is unstable.

Model

Diffusion (random walk) on graphs

Space: Simple connected graph $\mathcal{G} = (N, E)$.
$N = \{1, 2, \ldots, k\}$, $\sigma_j = \big|\{\{i,j\} \in E : i \in N\}\big|$ . . . degree of the node j

Process: A particle in a node may choose randomly a neighbour node and move to it during the unit time interval.

$$\{i,j\} \in E \Rightarrow \frac{1}{\sigma_j} = \Pr(\text{the particle enters the node } i \,|\, \text{it leaves the node } j)$$

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Adjacency matrix A, $a_{ij} = \begin{cases} 1, & \{i,j\} \in E, \\ 0, & \text{otherwise.} \end{cases}$

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$$\Delta \boldsymbol{x} = d(\mathsf{K} - \mathsf{I})\boldsymbol{x}.$$

Properties of matrices

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---

- D is left stochastic matrix: $\mathbf{1}^\mathsf{T}\mathsf{D} = \mathbf{1}^\mathsf{T}\big(\mathsf{I} - d(\mathsf{I}-\mathsf{K})\big) = \mathbf{1}^\mathsf{T} - d\mathbf{1}^\mathsf{T} + d\mathbf{1}^\mathsf{T}\mathsf{K} = \mathbf{1}^\mathsf{T}$
- λ is eigenvalue of matrix K with respective eigenvector $\boldsymbol{w}$ $\Leftrightarrow$
  $1 + (d - \lambda)$ is eigenvalue of matrix D with respective eigenvector $\boldsymbol{w}$:

$$\begin{aligned} \mathsf{K}\boldsymbol{w} &= \lambda\boldsymbol{w} \\ \mathsf{I}\boldsymbol{w} - d\mathsf{I}\boldsymbol{w} + d\mathsf{K}\boldsymbol{w} &= \boldsymbol{w} - d\boldsymbol{w} + d\lambda\boldsymbol{w} \\ \big(\mathsf{I} - d(\mathsf{I}-\mathsf{K})\big)\boldsymbol{w} &= \big(1 - d(1-\lambda)\big)\boldsymbol{w} \end{aligned}$$

Reaction in a node

$$x_i(t+1) = f\big(x_i(t)\big)$$

Reaction and diffusion on the graph

Reaction in a node,

$$x_i(t+\vartheta) = f\big(x_i(t)\big), \quad 0 < \vartheta \ll 1,$$

is followed by diffusion on the graph,

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$$\boldsymbol{F}(\boldsymbol{x}) := \begin{pmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_k) \end{pmatrix}$$

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Equilibrium and stability

$$\boldsymbol{x}(t+1) = \mathsf{D}\boldsymbol{F}\big(\boldsymbol{x}(t)\big), \qquad \mathsf{D} = \mathsf{I} - d(\mathsf{I} - \mathsf{K})$$

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Let $f(x^*) = x^*$.

If $\mathsf{K} = \mathsf{K}^\mathsf{T}$, then $\boldsymbol{x^*} = x^*\mathbf{1}$ is equilibrium of the reaction-diffusion equation.
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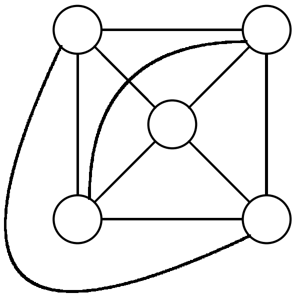
$\varrho\big(f'(x^*)\mathsf{D}\big) = f'(x^*)\varrho(\mathsf{D}) = f'(x^*)\big(1 + d(\varrho(\mathsf{K}) - 1)\big) < 1$

Graphs with $K = K^T$

Complete graph:



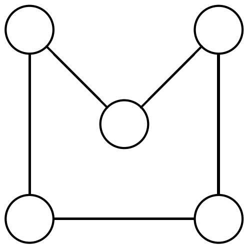
$$\mathsf{A} = \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{pmatrix}, \quad \mathsf{K} = \begin{pmatrix} 0 & \frac{1}{k-1} & \frac{1}{k-1} & \dots & \frac{1}{k-1} \\ \frac{1}{k-1} & 0 & \frac{1}{k-1} & \dots & \frac{1}{k-1} \\ \frac{1}{k-1} & \frac{1}{k-1} & 0 & \dots & \frac{1}{k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{k-1} & \frac{1}{k-1} & \frac{1}{k-1} & \dots & 0 \end{pmatrix}.$$

Eigenvalue $\lambda_1 = 1$, eigenvector $\mathbf{1}$,

eigenvalue $\lambda_2 = \frac{1}{1-k}$, eigenvectors $\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \\ -1 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ -1 \end{pmatrix}.$

Graphs with $\mathsf{K} = \mathsf{K}^T$

Cycle graph:



$$\mathsf{A} = \begin{pmatrix} 0 & 1 & 0 & \dots & 1 \\ 1 & 0 & 1 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad \mathsf{K} = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \dots & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & \dots & 0 \\ 0 & \frac{1}{2} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2} & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Eigenvalue $\lambda_1 = 1$, eigenvector $\mathbf{1}$,

eigenvalues $\lambda_j = \cos\frac{2(j-1)}{k}\pi$,

$$\text{eigenvectors} \begin{pmatrix} 1 \\ \cos\frac{2(j-1)}{k}\pi \\ \cos\frac{2(j-1)2}{k}\pi \\ \vdots \\ \cos\frac{2(j-1)(k-1)}{k}\pi \end{pmatrix}, \begin{pmatrix} 0 \\ \sin\frac{2(j-1)}{k}\pi \\ \sin\frac{2(j-1)2}{k}\pi \\ \vdots \\ \sin\frac{2(j-1)(k-1)}{k}\pi \end{pmatrix}, \; j = 2, 3, \dots, \left[\frac{k+1}{2}\right],$$

$$\lambda_{1+k/2} = -1, \text{ eigenvector } \begin{pmatrix} 1 \\ -1 \\ 1 \\ \vdots \\ -1 \end{pmatrix} \text{ for } k \text{ even.}$$

Result

Two component reaction

$$x(t+1) = f\big(x(t), y(t)\big)$$
$$y(t+1) = g\big(x(t), y(t)\big)$$

$x = x(t)$, $y = y(t)$ ... amount of the first and of the second component

Assumption: There is a non-trivial asymptotically stable equilibrium x^*, y^*.

Two component reaction

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$x = x(t)$, $y = y(t)$ ... amount of the first and of the second component

Assumption: There is a non-trivial asymptotically stable equilibrium x^*, y^*.

In details:

- $f(x^*, y^*) = x^* > 0$, $g(x^*, y^*) = y^* > 0$
- $|\operatorname{tr} \mathsf{B}| - 1 < \det \mathsf{B} < 1$

where

$$\mathsf{B} = \begin{pmatrix} f_x^* & f_y^* \\ g_x^* & g_y^* \end{pmatrix}$$

$$f_x^* := \frac{\partial f}{\partial x}(x^*, y^*), \quad f_y^* := \frac{\partial f}{\partial y}(x^*, y^*), \quad g_x^* := \frac{\partial g}{\partial x}(x^*, y^*), \quad g_y^* := \frac{\partial g}{\partial y}(x^*, y^*)$$

Two component reaction and diffusion

$x_i = x_i(t)$, $y_i = y_i(t)$ . . . amount of the first and of the second component in the node i at time t

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$d_1 = \Pr$(particle "x" leaves its node), $d_2 = \Pr$(particle "y" leaves its node)

$$\mathsf{D}_p = \mathsf{I} - d_p(\mathsf{I} - \mathsf{K}), \qquad \boldsymbol{F}(\boldsymbol{x},\boldsymbol{y}) = \begin{pmatrix} f(x_1,y_1) \\ f(x_2,y_2) \\ \vdots \\ f(x_k,y_k) \end{pmatrix}, \quad \boldsymbol{G}(\boldsymbol{x},\boldsymbol{y}) = \begin{pmatrix} g(x_1,y_1) \\ g(x_2,y_2) \\ \vdots \\ g(x_k,y_k) \end{pmatrix}.$$

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$$\boldsymbol{x}(t+1) = \mathsf{D}_1 \boldsymbol{F}\big(\boldsymbol{x}(t),\boldsymbol{y}(t)\big)$$
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Stability of equilibrium

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If $\mathsf{K} = \mathsf{K}^\mathsf{T}$ then $\begin{pmatrix} \boldsymbol{x}^* \\ \boldsymbol{y}^* \end{pmatrix} = \begin{pmatrix} x^* \mathbf{1} \\ y^* \mathbf{1} \end{pmatrix}$ is equilibrium of the equation.

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Linearization:

$$\mathsf{J}\left[\begin{pmatrix} \mathsf{D}_1 & \mathsf{O} \\ \mathsf{O} & \mathsf{D}_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{F}\big(\boldsymbol{x}^*, \boldsymbol{y}^*\big) \\ \boldsymbol{G}\big(\boldsymbol{x}^*, \boldsymbol{y}^*\big) \end{pmatrix}\right] = \begin{pmatrix} \mathsf{D}_1 & \mathsf{O} \\ \mathsf{O} & \mathsf{D}_2 \end{pmatrix} \begin{pmatrix} f_x^* \mathsf{I} & f_y^* \mathsf{I} \\ g_x^* \mathsf{I} & g_y^* \mathsf{I} \end{pmatrix}$$

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Evolution of deviations from the equilibrium:

$$\begin{pmatrix} \boldsymbol{u}(t) \\ \boldsymbol{v}(t) \end{pmatrix} := \begin{pmatrix} \boldsymbol{x}(t) \\ \boldsymbol{y}(t) \end{pmatrix} - \begin{pmatrix} \boldsymbol{x}^* \\ \boldsymbol{y}^* \end{pmatrix}, \qquad \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{pmatrix}(t+1) = \begin{pmatrix} \mathsf{D}_1 & \mathsf{O} \\ \mathsf{O} & \mathsf{D}_2 \end{pmatrix} \begin{pmatrix} f_x^*\boldsymbol{u}(t) + f_y^*\boldsymbol{v}(t) \\ g_x^*\boldsymbol{u}(t) + g_y^*\boldsymbol{v}(t) \end{pmatrix}$$

Stability of equilibrium

Linearised system:

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We search the solution in the form

$$\boldsymbol{u}(t) = \sum_{p=1}^{k} \alpha_p(t) \boldsymbol{w}_p, \quad \boldsymbol{v}(t) = \sum_{p=1}^{k} \beta_p(t) \boldsymbol{w}_p,$$

where $\boldsymbol{w}_p$ is the eigenvector corresponding to the eigenvalue λ_p of the matrix K.

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$$\begin{pmatrix} \mathsf{D}_1 & \mathsf{O} \\ \mathsf{O} & \mathsf{D}_2 \end{pmatrix} \begin{pmatrix} f_x^* \boldsymbol{u}(t) + f_y^* \boldsymbol{v}(t) \\ g_x^* \boldsymbol{u}(t) + g_y^* \boldsymbol{v}(t) \end{pmatrix} = \begin{pmatrix} \mathsf{D}_1 & \mathsf{O} \\ \mathsf{O} & \mathsf{D}_2 \end{pmatrix} \begin{pmatrix} \sum \big(f_x^* \alpha_p(t) + f_y^* \beta_p(t)\big) \boldsymbol{w}_p \\ \sum \big(g_x^* \alpha_p(t) + g_y^* \beta_p(t)\big) \boldsymbol{w}_p \end{pmatrix} =$$

$$= \begin{pmatrix} \sum \big(f_x^* \alpha_p(t) + f_y^* \beta_p(t)\big) \mathsf{D}_1 \boldsymbol{w}_p \\ \sum \big(g_x^* \alpha_p(t) + g_y^* \beta_p(t)\big) \mathsf{D}_2 \boldsymbol{w}_p \end{pmatrix} = \begin{pmatrix} \sum \big(1 + d_1(\lambda_p - 1)\big)\big(f_x^* \alpha_p(t) + f_y^* \beta_p(t)\big) \boldsymbol{w}_p \\ \sum \big(1 + d_2(\lambda_p - 1)\big)\big(g_x^* \alpha_p(t) + g_y^* \beta_p(t)\big) \boldsymbol{w}_p \end{pmatrix}$$

Linearised system:

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where $\boldsymbol{w}_p$ is the eigenvector corresponding to the eigenvalue λ_p of the matrix K.

Hence

$$\sum_{p=1}^{k} \alpha_p(t+1) \boldsymbol{w}_p = \sum_{p=1}^{k} \big(1 + d_1(\lambda_p - 1)\big)\big(f_x^* \alpha_p(t) + f_y^* \beta_p(t)\big) \boldsymbol{w}_p,$$

$$\sum_{p=1}^{k} \beta_p(t+1) \boldsymbol{w}_p = \sum_{p=1}^{k} \big(1 + d_2(\lambda_p - 1)\big)\big(g_x^* \alpha_p(t) + g_y^* \beta_p(t)\big) \boldsymbol{w}_p$$

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Consequently:

$$\begin{pmatrix} \alpha_p \\ \beta_p \end{pmatrix}(t+1) = \begin{pmatrix} \big(1 + d_1(\lambda_p - 1)\big) f_x^* & \big(1 + d_1(\lambda_p - 1)\big) f_y^* \\ \big(1 + d_2(\lambda_p - 1)\big) g_x^* & \big(1 + d_2(\lambda_p - 1)\big) g_y^* \end{pmatrix} \begin{pmatrix} \alpha_p \\ \beta_p \end{pmatrix}(t),$$

$$p = 1, 2, \ldots, k.$$

Stability of equilibrium

Linear 2-dimensional system

$$\begin{pmatrix}\alpha_p\\ \beta_p\end{pmatrix}(t+1) = \begin{pmatrix}\big(1+d_1(\lambda_p-1)\big)f_x^* & \big(1+d_1(\lambda_p-1)\big)f_y^*\\ \big(1+d_2(\lambda_p-1)\big)g_x^* & \big(1+d_2(\lambda_p-1)\big)g_y^*\end{pmatrix}\begin{pmatrix}\alpha_p\\ \beta_p\end{pmatrix}(t)$$

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$$\mathsf{C}:=\begin{pmatrix}\big(1+d_1(\lambda_p-1)\big)f_x^* & \big(1+d_1(\lambda_p-1)\big)f_y^*\\ \big(1+d_2(\lambda_p-1)\big)g_x^* & \big(1+d_2(\lambda_p-1)\big)g_y^*\end{pmatrix}$$

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Criterion of instability:

$$|\operatorname{tr}\mathsf{C}|>\det\mathsf{C}+1 \quad\text{or}\quad \det\mathsf{C}>1 \quad\text{or}\quad |\operatorname{tr}\mathsf{C}|>2,$$

where

$$\operatorname{tr}\mathsf{C}=f_x^*+g_y^*+(\lambda_p-1)(d_1f_x^*+d_2g_y^*)=(\lambda_p-1)(d_1f_x^*+d_2g_y^*)+\operatorname{tr}\mathsf{B},$$

$$\begin{aligned}\det\mathsf{C}&=\big(d_1d_2(\lambda_p-1)^2+(d_1+d_2)(\lambda_p-1)+1\big)(f_x^*g_y^*-f_y^*g_x^*)=\\ &=\big(1+(d_1+d_2)(\lambda_p-1)+d_1d_2(\lambda_p-1)^2\big)\det\mathsf{B}.\end{aligned}$$

The main result

Let f, g be the functions such that $f(x^*, y^*) = x^*$, $g(x^*, y^*) = y^*$ and

$$\mathsf{B} = \begin{pmatrix} f_x(x^*, y^*) & f_y(x^*, y^*) \\ g_x(x^*, y^*) & g_y(x^*, y^*) \end{pmatrix}, \qquad |\operatorname{tr} \mathsf{B}| - 1 < \det \mathsf{B} < 1.$$

Let $\mathsf{K} = \mathsf{K}^\mathsf{T}$ be stochastic matrix, d_1, d_2 numbers such that $0 < d_1, d_2 \leq 1$. Put $\mathsf{D}_j = \mathsf{I} - d_j(\mathsf{I} - \mathsf{K})$, $j = 1, 2$.
If there is an eigenvalue λ_p of the matrix K such that

$$\left|\operatorname{tr} \mathsf{B} + (\lambda_p - 1)\big(d_1 f_x(x^*, y^*) + d_2 g_y(x^*, y^*)\big)\right| > 1 + \big(d_1 d_2 (\lambda_p - 1)^2 + (d_1 + d_2)(\lambda_p - 1) + 1\big) \det \mathsf{B},$$

or

$$\big(d_1 d_2 (\lambda_p - 1)^2 + (d_1 + d_2)(\lambda_p - 1) + 1\big) \det \mathsf{B} > 2,$$

or

$$\left|\operatorname{tr} \mathsf{B} + (\lambda_p - 1)\big(d_1 f_x(x^*, y^*) + d_2 g_y(x^*, y^*)\big)\right| > 2,$$

then the equilibrium $\boldsymbol{x^*} \equiv x^* \mathbf{1}$, $\boldsymbol{y^*} \equiv y^* \mathbf{1}$ of the system

$$\begin{pmatrix} x_1(t+1) \\ x_2(t+1) \\ \vdots \\ x_k(t+1) \end{pmatrix} = \mathsf{D}_1 \begin{pmatrix} f\big(x_1(t), y_1(t)\big) \\ f\big(x_2(t), y_2(t)\big) \\ \vdots \\ f\big(x_k(t), y_k(t)\big) \end{pmatrix}, \qquad \begin{pmatrix} y_1(t+1) \\ y_2(t+1) \\ \vdots \\ y_k(t+1) \end{pmatrix} = \mathsf{D}_2 \begin{pmatrix} g\big(x_1(t), y_1(t)\big) \\ g\big(x_2(t), y_2(t)\big) \\ \vdots \\ g\big(x_k(t), y_k(t)\big) \end{pmatrix}$$

is unstable.

Example

$$f(x,y) = r_1 x \exp\left(1 - \frac{x}{K_1} + \gamma_{12} y\right), \quad g(x,y) = r_2 y \exp\left(1 - \frac{y}{K_2} + \gamma_{21} x\right)$$

$r_1 = 1.4$, $r_2 = 0.7$, $K_1 = 6$, $K_2 = 1$, $\gamma_{12} = -0.5$, $\gamma_{21} = 1.9$,
i.e. $x^* = 0.91$, $y^* = 2.37$, $\operatorname{tr} \mathsf{B} = -0.52$; $\quad |\operatorname{tr} \mathsf{B}| - 1 = -0.48$, $\det \mathsf{B} = 0.88$,
Hence, the reaction equilibrium is stable.

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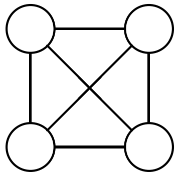

$$d_1 = 0.1, \quad d_2 = 0.8, \qquad \mathsf{K} = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}$$

$\lambda_2 = -\frac{1}{3},$

$|\operatorname{tr} \mathsf{C}| - 1 = -0.173 < -0.051 = \det \mathsf{C} < 1 \Rightarrow$ reaction equilibrium remains stable.

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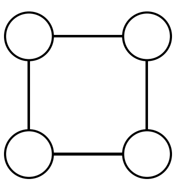

$$d_1 = 0.1, \quad d_2 = 0.8, \qquad \mathsf{K} = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}$$

$\lambda_3 = -1$,

$|\operatorname{tr} \mathsf{C}| - 1 = 0.501 > -0.424 = \det \mathsf{C} \Rightarrow$ reaction equilibrium is unstable.

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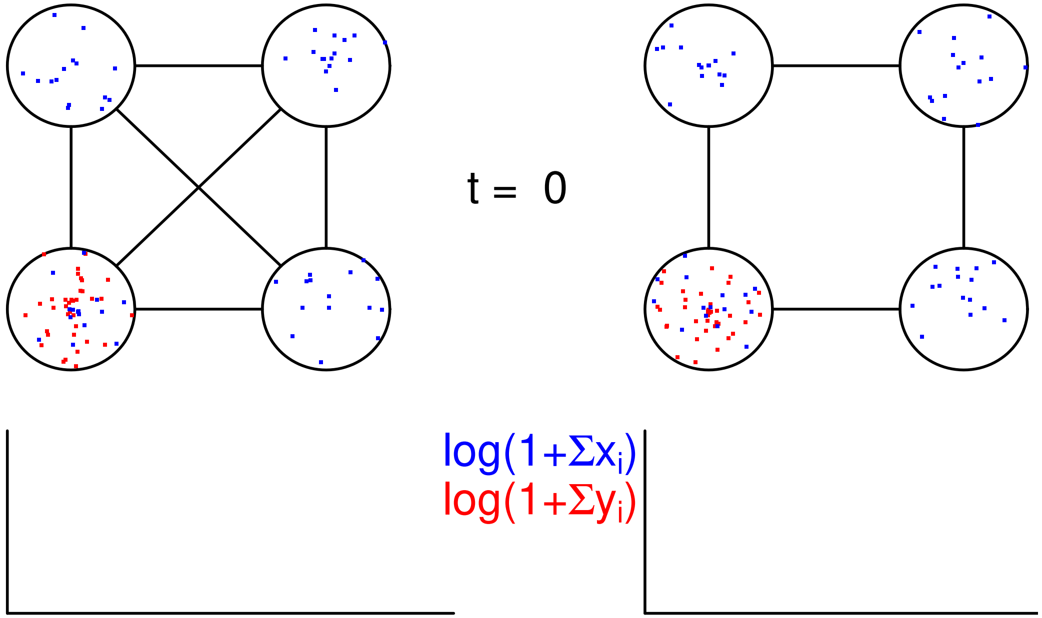
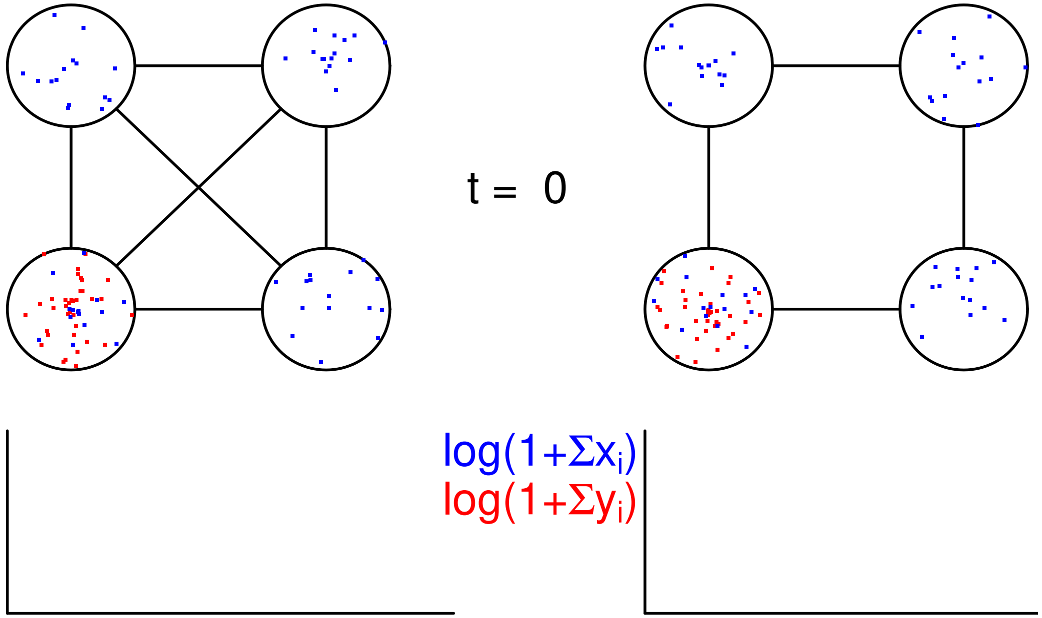
$d_1 = 0.1$, $d_2 = 0.8$.



$$f(x,y) = r_1 x \exp\left(1 - \frac{x}{K_1} + \gamma_{12} y\right), \quad g(x,y) = r_2 y \exp\left(1 - \frac{y}{K_2} + \gamma_{21} x\right)$$

$r_1 = 1.4$, $r_2 = 0.7$, $K_1 = 6$, $K_2 = 1$, $\gamma_{12} = -0.5$, $\gamma_{21} = 1.9$,
$d_1 = 0.1$, $d_2 = 0.8$.



Thank you