

# Singular solutions of Protter problems for the wave equation

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## Mixed-Type Equations and Transonic Flows.

Transonic potential flows in the fluid dynamics lead to boundary value problems for equations of mixed type.

At a certain point of the airfoil, the speed of the flow exceeds the speed of sound and a shock wave is formed.

Across the shock there is a rapid rise in pressure, temperature and density.

Using the hodograph variables for 2-D flow, the typical equation is:

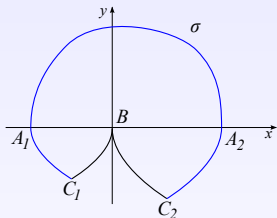
$$K(y)u_{xx} + u_{yy} = 0$$

Subsonic flow:  $K(y) > 0 \Rightarrow$  elliptic equation

Supersonic flow:  $K(y) < 0 \Rightarrow$  hyperbolic equation

## Guderley-Morawetz Problem.

The Guderley-Morawetz problem is connected to the models of flows around airfoils.



Equation of mixed elliptic-hyperbolic type

$$K(y)u_{xx} + u_{yy} = 0,$$

where  $K(y)y > 0$  for  $y \neq 0$ .

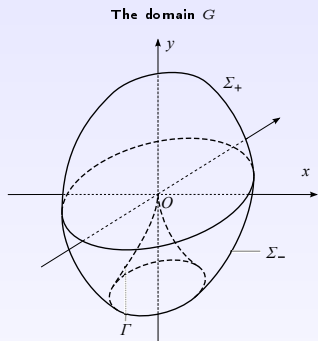
Boundary conditions on  $\sigma$ ,  $A_1C_1$  and  $A_2C_2$ .

The plane Guderley-Morawetz mixed-type problem is well studied – see the survey by Morawetz (2004) for the classical 2-D mixed type BVPs and their transonic background.

Morawetz (1958): Existence of weak solutions and uniqueness of strong solutions in weighted Sobolev spaces.

Lax and Phillips (1960): The weak solutions are strong. Regularity.

## Protter-Morawetz Problem.



In the 1950s M.H. Protter proposed multidimensional analogues of the 2-D Guderley-Morawetz problem.

Let  $K(y)$  is such that  $yK(y) > 0$  for  $y \neq 0$ . In  $R^4$  with points  $(x, y) = (x_1, x_2, x_3, y)$  consider in  $G$  the equation

$$K(y)(u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3}) - u_{yy} = f(x, y)$$

with boundary conditions  $u|_{\Sigma_+ \cap \Sigma_-} = 0$ .

The Protter-Morawetz problems have been studied by many authors in 1970s and 1980s, but a general understanding of the situation is still not at hand. Even the question of well posedness is surprisingly subtle and not completely resolved. Uniqueness results for quasiregular solutions were obtained by Aziz and Schneider (1979), but there are real obstructions to existence in this class.

To explain the difficulties and illustrate the differences with the 2-D case we study the Protter problems in the hyperbolic part of the domain.

## Protter Problems.

Consider the wave equation in  $\mathbb{R}^4$

$$u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3} - u_{tt} = f(x, t)$$

with points  $(x, t) = (x_1, x_2, x_3, t)$  in the domain

$$\Omega = \{(x, t) : 0 < t < 1/2, t < \sqrt{x_1^2 + x_2^2 + x_3^2} < 1 - t\},$$

bounded by the two characteristic cones

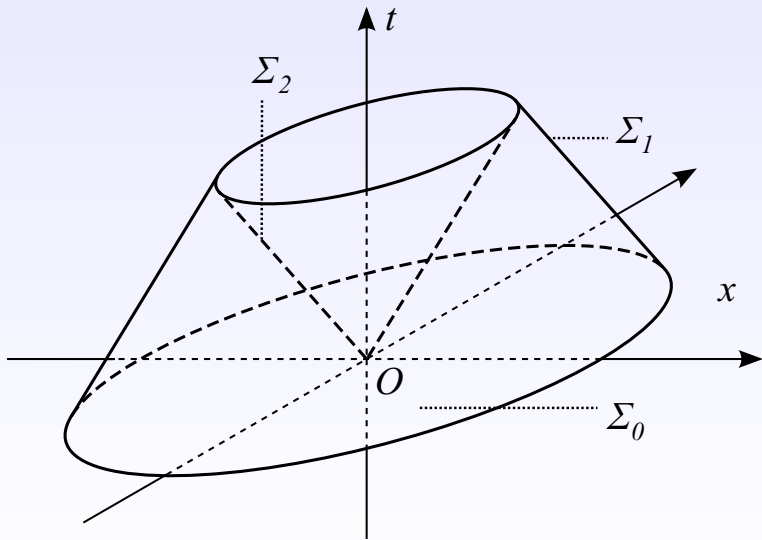
$$\Sigma_1 = \{(x, t) : 0 < t < 1/2, \sqrt{x_1^2 + x_2^2 + x_3^2} = 1 - t\},$$

$$\Sigma_2 = \{(x, t) : 0 < t < 1/2, \sqrt{x_1^2 + x_2^2 + x_3^2} = t\}$$

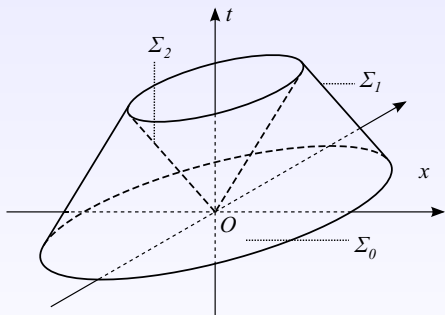
and the ball

$$\Sigma_0 = \{t = 0, \sqrt{x_1^2 + x_2^2 + x_3^2} < 1\},$$

centered at the origin  $O : x = 0, t = 0$ .



The following multidimensional analogues of Darboux problems were proposed by Murray Protter:



**Problem P1.** Find a solution of the wave equation in  $\Omega$  which satisfies the boundary conditions

$$u|_{\Sigma_0} = 0, \quad u|_{\Sigma_1} = 0.$$

**Problem P1\*.** Find a solution of the wave equation in  $\Omega$  which satisfies the adjoint boundary conditions

$$u|_{\Sigma_0} = 0, \quad u|_{\Sigma_2} = 0.$$

Let us define for  $k \in \mathbb{N} \cup \{0\}$  the functions

$$h_k(\xi, \eta) = \int_{\eta}^{\xi} s^k P_n \left( \frac{\xi\eta + s^2}{s(\xi + \eta)} \right) ds.$$

where  $P_n$  are the *Legendre polynomials*, defined by the Rodrigues formula:

$$P_n(s) := \frac{1}{2^n n!} \frac{d^n}{ds^n} (s^2 - 1)^n$$

### Lemma

*The functions*

$$v_{k,m}^n(x, t) = |x|^{-1} h_{n-2k-2} \left( \frac{|x|+t}{2}, \frac{|x|-t}{2} \right) Y_n^m(x).$$

are classical solutions from  $C^\infty(\Omega) \cap C(\overline{\Omega})$  of the homogeneous problem  $P1^*$  for  $n \in \mathbb{N}$ ,  $m = 1, \dots, 2n+1$  and  $k = 0, 1, \dots, [(n-1)/2] - 2$ .

Tong Kwang Chang (1957); Khe Kan Cher (1998)



## Spherical Functions.

One can define the spherical functions on the unit sphere  $S^2$  in  $\mathbb{R}^3$  by

$$Y_n^{2m}(x_1, x_2, x_3) = C_n^m \frac{d^m}{dx_3^m} P_n(x_3) \operatorname{Im} \{(x_2 + ix_1)^m\}, \text{ for } m = 1, \dots, n$$

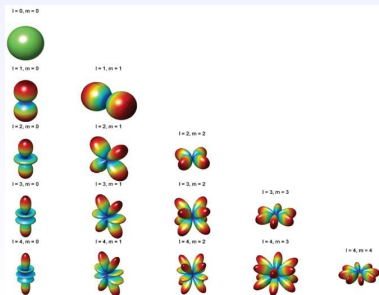
$$Y_n^{2m+1}(x_1, x_2, x_3) = C_n^m \frac{d^m}{dx_3^m} P_n(x_3) \operatorname{Re} \{(x_2 + ix_1)^m\}, \text{ for } m = 0, \dots, n,$$

where  $C_n^m$  are constants and  $P_n$  are the *Legendre polynomials*.

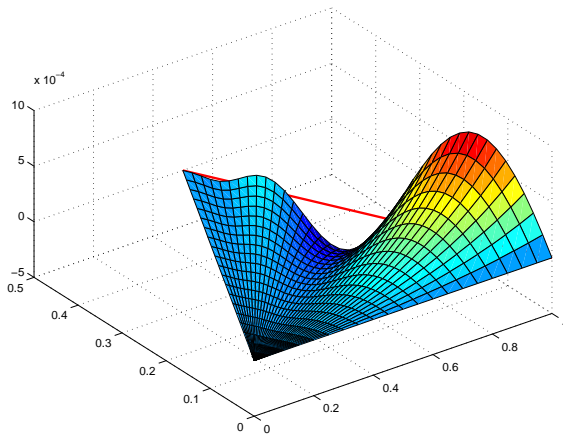
$\{Y_n^m\}$ :  $n \in \mathbb{N} \cup \{0\}$ ;  $m = 1, \dots, 2n + 1$   
is a complete orthonormal system in  $L^2(S^2)$ .

For convenience, we keep the same notation for the radial extension of the spherical function to  $\mathbb{R}^3 \setminus \{O\}$ , i.e.

$$Y_n^m(x) := Y_n^m(x/|x|).$$



$$r^{-1}h_{n-2k-2}\left(\frac{r+t}{2}, \frac{r-t}{2}\right) \text{ with } n = 11 \text{ and } k = 3.$$



Garabedian (1960) proved the uniqueness of the classical solution of Protter problem. However, a necessary condition for the existence of classical solution for the Problem  $P1$  is the orthogonality of the right-hand side function  $f$  to all functions  $v_{k,m}^n(x,t)$ .

To avoid an infinite number of necessary conditions, we introduce *generalized solutions* for the problem  $P1$ , eventually with a singularity at the origin  $O$ .

### Definition

A function  $u = u(x,t)$  is called a generalized solution of the problem  $P1$  in  $\Omega$ , if the following conditions are satisfied:

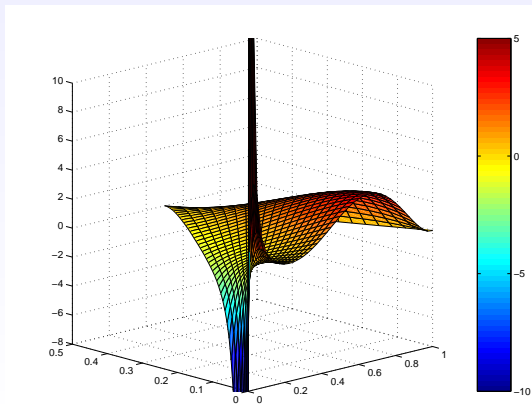
- 1)  $u \in C^1(\overline{\Omega} \setminus O)$ ,  $u|_{\Sigma_0 \setminus O} = 0$ ,  $u|_{\Sigma_1} = 0$ , and
- 2) the identity

$$\int_{\Omega} (u_t w_t - u_{x_1} w_{x_1} - u_{x_2} w_{x_2} - u_{x_3} w_{x_3} - f w) dx dt = 0$$

holds for all  $w \in C^1(\overline{\Omega})$  such that  $w = 0$  on  $\Sigma_0$  and in a neighborhood of  $\Sigma_2$ .

The singular solution of Protter problems were studied by: Popivanov and Schneider; Aldashev; korean mathematicians Jong Duek Jeon et al.(1996), Jong Bae Choi, Jong Yeoul Park (2002).

Popivanov and Schneider (1993) proved the uniqueness of the generalized solutions of Protter problems. It is shown that for each  $n \in \mathbb{N}$  there exists a right-hand side function  $f \in C^n(\bar{\Omega})$ , for which the generalized solution has a strong power-type singularity like  $r^{-n}$ . This singularity is isolated at the vertex  $O$  and does not propagate along the characteristic cone.



## Existence of bounded solutions.

N.Popivanov, T.Popov, R.Scherer

Protter-Morawetz multidimensional problems,

Proceedings of the Steklov Institute of Mathematics 278 Issue 1 (2012) 179-198.

### Theorem

*Let the function  $f(x, t)$  belong to  $C^{10}(\overline{\Omega})$ . Then the necessary and sufficient conditions for existence of bounded generalized solution  $u(x, t)$  of the Protter Problem P1 are*

$$\int_{\Omega} v_{k,m}^n(x, t) f(x, t) dx dt = 0,$$

*for all  $n \in \mathbb{N}$ ,  $k = 0, \dots, [\frac{n-1}{2}]$ ,  $m = 1, \dots, 2n + 1$ .*

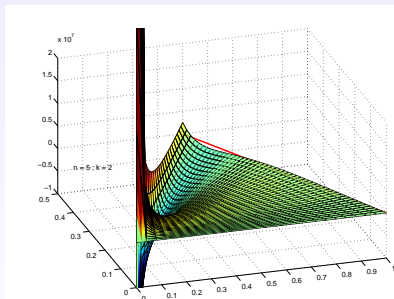
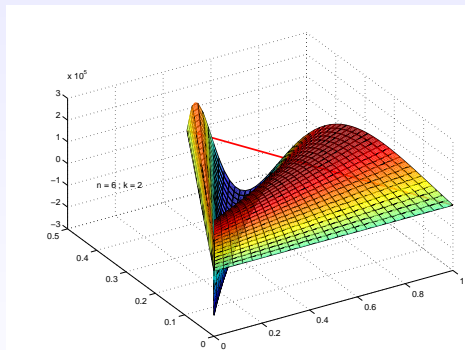
*Moreover, this generalized solution  $u(x, t) \in C^1(\overline{\Omega} \setminus O)$  and satisfies the a priori estimates*

$$|u(x, t)| \leq C \|f\|_{C^{10}(\overline{\Omega})};$$

$$\sum_{i=1}^3 |u_{x_i}(x, t)| + |u_t(x, t)| \leq C(|x|^2 + t^2)^{-1} \|f\|_{C^{10}(\overline{\Omega})}$$

*where the constant  $C$  is independent of the function  $f(x, t)$ .*

Naturally, the necessary orthogonality conditions for the existence of bounded solutions of Problem  $P1$  include the functions  $v_{k,m}^n$  from Lemma 1. However, it is interesting that there are also some others:



## Singular solutions of Problem P1.

Generally, a smooth function  $f(x, t)$  can be expanded as a harmonic series

$$f(x, t) = \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} f_n^m(|x|, t) Y_n^m(x)$$

with Fourier coefficients

$$f_n^m(r, t) := \int_{S(r)} f(x, t) Y_n^m(x) d\sigma_r,$$

where  $S(r)$  is the three-dimensional sphere  $\{x \in \mathbb{R}^3 : |x| = r\}$ .

The behaviour of the solution depends on the parameters

$$\beta_{k,m}^n := \int_{\Omega} v_{k,m}^n(x, t) f(x, t) dx dt,$$

where  $n = 0, \dots, l$ ;  $k = 0, \dots, \lfloor \frac{n-1}{2} \rfloor$  and  $m = 1, \dots, 2n + 1$ .

Let us introduce for  $p \in \mathbb{R}$  and  $k \in \mathbb{N}$  the series

$$\|f; n^p; C^k\| := \|f_0^0(|x|, t)\|_{C^0(\Omega)} + \sum_{n=1}^{\infty} n^p \left\| \sum_{m=1}^{2n+1} f_n^m(|x|, t) Y_n^m(x) \right\|_{C^k(\Omega)}$$

$$\Phi(s) := \sum_{n=1}^{\infty} \left[ \sum_{m=1}^{2n+1} \sum_{k=0}^{[n/2]} |\beta_{k,m}^n| \right] s^n.$$

### Theorem

*Suppose:  $f(x, t) \in C^1(\overline{\Omega})$ ; the series  $\|f; n^6; C^0\|$  and  $\|f; n^4; C^1\|$  are convergent; the power series  $\Phi(s)$  has an infinite radius of convergence. Then there exists a unique generalized solution  $u(x, t) \in C^1(\overline{\Omega} \setminus O)$  of Problem P1, and*

$$|u(x, t)| \leq C \left[ \Phi \left( \frac{C_1}{|x| + t} \right) + \|f; n^6; C^0\| + \|f; n^4; C^1\| \right];$$

$$\sum_{i=1}^3 |u_{x_i}(x, t)| + |u_t(x, t)| \leq C|x|^{-2} \left[ \Phi \left( \frac{C_2}{|x| + t} \right) + \|f; n^6; C^0\| \right];$$

where the constants  $C$ ,  $C_1$  and  $C_2$  are independent of  $f(x, t)$ .



Next, we compare the situation here with the results of Popivanov & Schneider (1995) for (2+1)-D Protter Problems.

The sufficient condition for existence of generalized solution in the (2+1)-D case is the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{n} I_0 \left( \frac{2n}{\varepsilon} \right) \left( \|f_n^1\|_{C^0(\Omega)} + \|f_n^2\|_{C^0(\Omega)} \right), \text{ for all } \varepsilon > 0,$$

where  $f_n^i$  are the Fourier coefficients for the right-hand side (the analogues of  $f_n^m$  here).

The function  $I_0$  is the modified Bessel function of first kind:

$$I_0(s) := \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left( \frac{s}{2} \right)^{2k}.$$

We can use the estimate

$$I_0(s) \leq e^s \text{ for } s \geq 0.$$

### (2+1)-D case

Suppose that the power series

$$\Phi_1(s) := \sum_{n=1}^{\infty} n^{-1} \left( \|f_n^1\|_{C^0(\Omega)} + \|f_n^2\|_{C^0(\Omega)} \right) s^n$$

has an infinite radius of convergence.

Then there exist unique generalized solution  $u$ , and near the origin we have the estimate

$$|u(x, t)| \leq C \Phi_1 \left[ \exp \left( \frac{2}{|x|} \right) \right].$$

where the constant  $C$  is independent of  $f$ .

### (3+1)-D case

Suppose that the power series

$$\Phi_2(s) := \sum_{n=1}^{\infty} \left[ \sum_{m=1}^{2n+1} \|f_n^m\|_{C^0(\Omega)} \right] s^n$$

has an infinite radius of convergence.

Then there exist unique generalized solution  $u$ , and near the origin we have the estimate

$$|u(x, t)| \leq C \Phi_2 \left[ \frac{C_0}{|x| + t} \right].$$

where the constants  $C$  and  $C_0$  are independent of  $f$ .

## Construction of singular solutions

If  $f$  is a harmonic polynomial the solution can have only power type singularity. However, in the general case stronger singularities are possible.

Suppose that the power series with coefficients  $\alpha_p \geq 0$

$$\phi(s) := \sum_{p=0}^{\infty} \alpha_p s^p$$

has infinite radius of convergency.

Is there a solution with singularity at the origin like  $\phi(1/t)$ ?

Recall that the parameters

$$\beta_{k,m}^n := \int_{\Omega} v_{k,m}^n(x, t) f(x, t) dx dt$$

“control” the behaviour of the singularity of the solution.

## Theorem

Let  $f(x, t) \in C^1(\overline{\Omega})$ , the series  $\|f; n^6; C^0\|$ ,  $\|f; n^4; C^1\|$  are convergent, and the power series  $\Phi(s)$  has an infinite radius of convergence. Suppose that there is  $x^* = (x_1^*, x_2^*, x_3^*) \in \mathbb{R}^3$  such that

$$\sum_{k=0}^{\infty} \sum_{m=1}^{2p+4k+1} p a_{p+2k, 2k} \beta_{m, k}^{p+2k} Y_{p+2k}^m(x^*) \geq \alpha_p$$

Then there exist a number  $\delta \in (0, 1/2)$  that the unique generalized solution  $u(x, t)$  of Problem P1 satisfies the estimate

$$|u(tx_1^*, tx_2^*, tx_3^*, t)| \geq \phi \left( \frac{1}{2t} \right) \quad \text{for } t \in (0, \delta).$$

Here  $a_{n, 2k}$  are the coefficients of the Legendre polynomial:

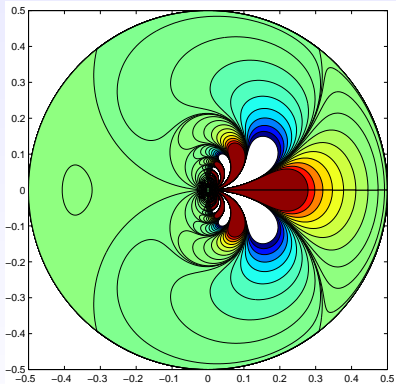
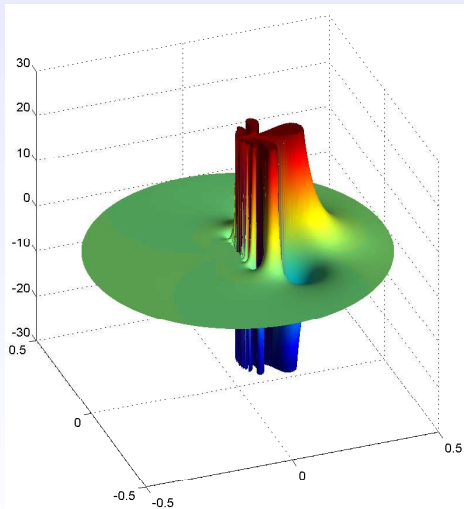
$$P_n(s) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} a_{n, 2k} s^{n-2k}, \quad a_{n, 2k} := (-1)^k \frac{(2n-2k)!}{2^n k! (n-k)! (n-2k)!}.$$

For example, it is possible to build an appropriate function  $f$  for the constants  $\alpha_p = \frac{1}{p!}$ , and thus the corresponding solution to grow at  $O$ .

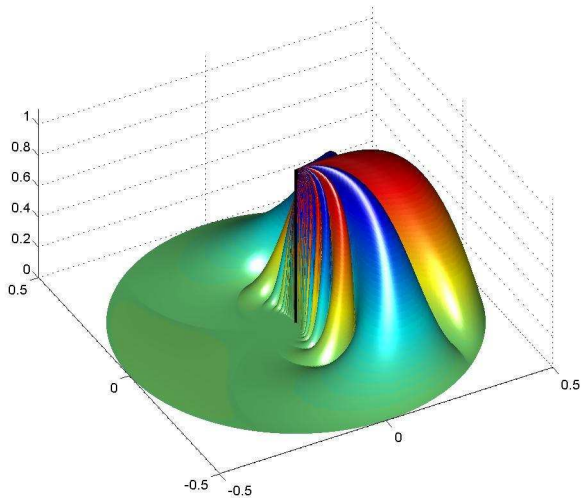
**THEOREM.** *There exists a function  $f \in C^\infty(\overline{\Omega})$  and a positive number  $\delta \in (0, 1/2)$ , such that the corresponding unique generalized solution  $u(x_1, x_2, x_3, t) \in C^1(\overline{\Omega} \setminus O)$  of the Problem P1 in  $\mathbb{R}^4$  with right-hand function  $f$ , satisfies the estimate*

$$u(0, 0, t, t) \geq \exp\left(\frac{1}{t}\right) \quad \text{for } 0 < t < \delta.$$

## Generalized solution with exponential growth



Graph of  $\frac{\ln(1+\ln(1+|u|))}{1-\ln(t)}$



$$u \sim \exp(C t^{-1})$$

THANK YOU FOR YOUR ATTENTION!