UNIVERSITÀ DEGLI STUDI DI TRIESTE

dipartimento di matematica e geoscienze

How to determine the shape of the human cornea: a contribution from nonlinear analysis

Pierpaolo Omari

Dipartimento di Matematica e Geoscienze
Università degli Studi di Trieste, Italy
E_mail: omari@units.it

DIFFERENTIAL EQUATIONS AND APPLICATIONS
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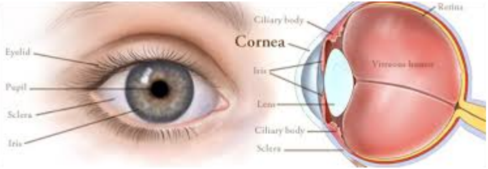
Joint research with: I. Coelho (ISEL), C. Corsato (UniTS), C. De Coster (UVHC), F. Obersnel (UniTS), A. Soranzo (UniTS)

Describing the geometry of the human cornea

- The cornea is the transparent front part of the eye that covers the iris, pupil, and anterior chamber: the cornea, with the anterior chamber and the lens, refracts light.
- In humans, the refractive power of the cornea is about 43 dioptres: the cornea accounts for approximately $2/3$ of the eye's total optical power.
- Better understanding its geometry and its mechanism may help to treat various common sight diseases, such as myopia, hyperopia, astigmatism. Indeed, refractive surgery and contact lens fitting depend on the accuracy of models describing corneal topography.



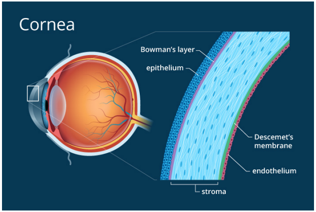
Models for the geometry of the human cornea

There are various mathematical models for the corneal geometry that are presently in use.

- The simplest ones are based on conic sections (parabolas, or ellipses) used as meridians for surfaces of revolution: they provide fairly good fitting results and are very easy to use, but without much physical motivation.
- There are models built on the theory of shells (i.e., solids thin in one direction) and finite element methods: they are more accurate, but an high price of complexity must be paid.
- Other popular models are based on orthogonal polynomials, or on special functions: they are used to describe aberrations in lens and cornea.

Cornea's typical sizes:

- eye's size 24 mm
- corneal diameters: 11.7 mm, the largest,
  10.6 mm, the smallest
- corneal thikness: 0.5 — 0.6 mm, in the center,
  0.6 — 0.8 mm, at the periphery
- cornea consists of 5 layers.



A new model for the geometry of the human cornea

In 2013 Okrasinski & Płociniczak introduced a new nonlinear model, although in their works they discussed a partial linearization that, however, revealed effective from the numerical point of view.

Main assumptions

- cornea is a thin elastic membrane
- u denotes the height of the corneal surface over a reference plane region Ω (an ellipsis), and the surface is kept fixed at the boundary $\partial\Omega$;
- three forces act over: the surface tension, of modulus T, a restoring force, of elastic constant k, a force associated with the intraocular pressure P;
- balancing forces yields

$$\begin{cases} T\mathrm{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) - ku + \dfrac{P}{\sqrt{1+|\nabla u|^2}} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Prescribed anisotropic mean curvature equation

Rearranging terms we are led to the equation

$$(E) \qquad -\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = -au + \frac{b}{\sqrt{1+|\nabla u|^2}} \quad \text{in } \Omega$$

- Ω bounded domain in $\mathbb{R}^N$, with boundary $\partial\Omega \in C^{0,1}$
- $a > 0$ and $b > 0$ parameters.

Formally, (E) is the Euler-Lagrange equation of the functional

$$\int_\Omega e^{-bu}\sqrt{1+|\nabla u|^2} - \frac{b}{a}\int_\Omega e^{-bu}\left(u + \frac{1}{b}\right),$$

involving the anisotropic area term $\int_\Omega e^{-bu}\sqrt{1+|\nabla u|^2}$.

Remark Equation (E) also provides a model for describing capillarity phenomena for compressible fluids (Finn, 2001).

Aims

I want to discuss

- existence
- uniqueness
- regularity
- boundary behaviour
- stability
- structure of the set

of the solutions of the Dirichlet problem

$$(P) \quad \begin{cases} -\mathrm{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = -au + \dfrac{b}{\sqrt{1+|\nabla u|^2}} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Hurdles

The mean curvature operator $\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)$ is

- non-uniformly elliptic: degenerate
- non-homogeneous: $\sim \Delta_2 u = \operatorname{div}(\nabla u)$ at 0,

  $\sim \Delta_1 u = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)$ at ∞.

This may yield non-existence phenomena and loss of regularity.

The apparently harmless term $\frac{b}{\sqrt{1+|\nabla u|^2}}$ forces to replace

the usual area functional $\int_\Omega \sqrt{1+|\nabla u|^2}$

with the anisotropic area functional $\int_\Omega e^{-bu}\sqrt{1+|\nabla u|^2}$.

Notion of solution

The presence of the mean curvature operator has a relevant impact on the morphology of solutions;
in general, one cannot expect that

- the solutions be regular
- the boundary conditions be attained.

Thus we need to introduce an appropriate notion of solution for the problem under consideration: it is related to the notion of

- pseudosolution (Temam, Lischnewski, 1971)
- weak/generalized solution (Giaquinta, Giusti, Miranda, 1974)

given for the minimal surface equation, or for a class of prescribed mean curvature equations, respectively.
Our definition is somehow implicit in a work of Lischnewski, 1978.

Notions of solution

Generalized solution

u is a generalized solution if

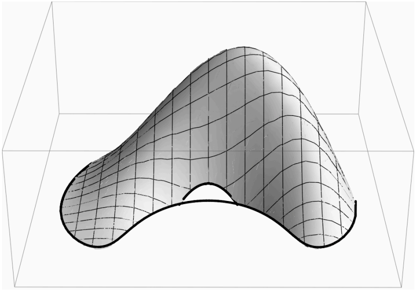
- $u \in W^{1,1}(\Omega)$ and $\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \in L^N(\Omega)$

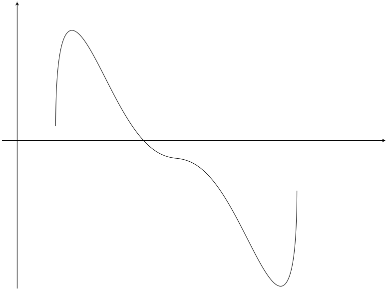
  $$\left(\Longrightarrow \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \in X(\Omega)_N \Longrightarrow \left[\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}, \nu\right] \in L^\infty(\partial\Omega) \right)$$

- u satisfies
  - a.e. in Ω

    $$-\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = -au + \frac{b}{\sqrt{1+|\nabla u|^2}}$$

  - $\mathcal{H}^{N-1}$-a.e. on $\partial\Omega$
    - either $u(x) = 0$
    - or $u(x) > 0$ and $\left[\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}, \nu\right](x) = -1$
    - or $u(x) < 0$ and $\left[\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}, \nu\right](x) = 1.$





Graph of a 2-D generalized solution

1-D profile of a generalized solution

Notions of solution

Classical solution

- A generalized solution u is classical if $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ and $u = 0$ on $\partial\Omega$.

Singular solution

- A generalized solution u is singular if it is not classical.

Program

Discuss the existence of

1. generalized solutions
2. classical solutions
3. singular solutions

Classical solutions: uniqueness

A uniqueness result

- $\forall a \geq 0\ \forall b \in \mathbb{R}$ problem (P) has at most one classical solution.

Sketch of proof

- change of variable: $v = e^{-bu}$
- rewrite (P) as

$$(Q) \quad \begin{cases} -\mathrm{div}\left(\dfrac{\nabla v}{\sqrt{v^2 + b^{-2}|\nabla v|^2}}\right) = -a \ln v - \dfrac{b^2 v}{\sqrt{v^2 + b^{-2}|\nabla v|^2}} & \text{in } \Omega \\ v = 1 & \text{on } \partial\Omega \end{cases}$$

- recast (Q) as a variational inequality:

$$(VI) \quad \int_\Omega \sqrt{w^2 + b^{-2}|\nabla w|^2} - \int_\Omega \sqrt{v^2 + b^{-2}|\nabla v|^2} \geq -\frac{a}{b^2} \int_\Omega \ln v\,(w - v)$$
$$\forall w \in W^{1,1}(\Omega),\ w = 1 \text{ on } \partial\Omega,$$

- exploit convexity and monotonicity (of the 0-order term)

Existence of classical solutions

An immediate consequence

$\forall a \geq 0$ and $b = 0$, $u = 0$ is the unique solution of

$$(P) \quad \begin{cases} -\mathrm{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = -au + \dfrac{b}{\sqrt{1+|\nabla u|^2}} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

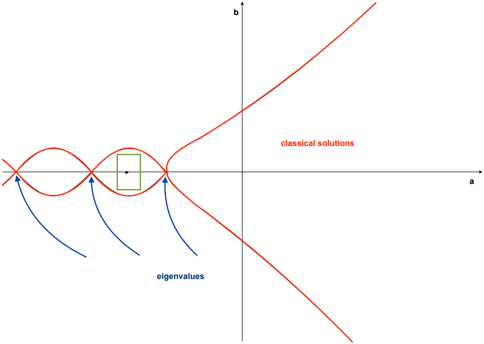
It is natural to infer, by perturbation, the existence of small classical solutions for small b.

Existence of classical solutions: a local result (b small)

Here: $\partial\Omega \in C^{2,\alpha}$.

Existence of small classical solutions via the IFT

$\forall a_0 \notin \Sigma = \mathrm{Spec}(\Delta, H_0^1(\Omega))$ $\exists \delta_0 > 0$ such that $|a - a_0| < \delta_0, |b| < \delta_0 \implies$ problem (P) has a unique solution $u = u(a, b) \in C^2(\overline{\Omega})$, which stems from the trivial solution.



Existence of classical solutions

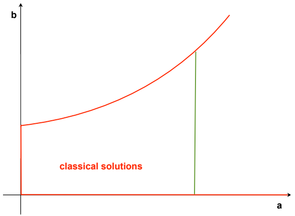
Existence of a maximal branch via the LS degree

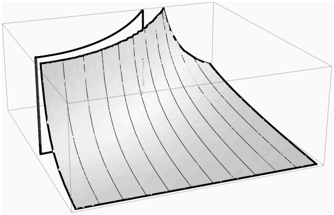
$\forall a > 0\ \exists b_\infty(a) \in]0, +\infty]$ such that, setting

$$\mathcal{E} = \bigcup_{a>0} (\{a\} \times]0, b_\infty(a)[) \subseteq \mathbb{R}_0^+ \times \mathbb{R}_0^+,$$

$\forall (a, b) \in \mathcal{E}$, problem (P) has a unique solution $u = u(a, b) \in C^2(\overline{\Omega})$, which is linearly stable, smoothly depends on the parameters (a, b) in the topology of $C^2(\overline{\Omega})$ and satisfies, in case $b_\infty(a) < +\infty$,

$$\|\nabla u(a, b)\|_\infty \to +\infty, \quad \text{as } b \to b_\infty(a).$$





Existence of classical solutions: global results

Question

What happens for b large ?

First step

Standard strategy: attack the problem by exploiting its symmetry properties.

Remark Problem (P) is invariant under orthogonal transformations: it is natural to look for radially symmetric solutions if

- $\Omega = B_R$ is a ball
- $\Omega = S_{r,R}$ is a spherical shell

Warning As we are going to see, in the two cases the solvability patters are quite different.

In fact, these two situations reveal paradigmatic for the whole discussion.

Existence of classical solutions: global results

Existence of classical radial solutions in balls

$\forall N \geq 1\ \forall a > 0\ \forall b > 0\ \forall R > 0$ problem (P) has a unique solution $u \in C^2(\overline{B}_R)$, such that $u(x) = v(|x|)$, with v positive, decreasing, concave in $[0, R]$.

Sketch of proof (Upper and lower solutions method)

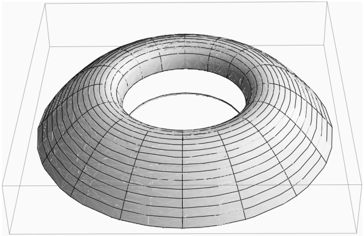
- $-v'' = -av(1 + v'^2)^{3/2} + b(1 + v'^2) + \frac{N-1}{t} v'(1 + v'^2) \quad$ in $]0, R[$, $v'(0) = 0, \; v(R) = 0$
- $\alpha = 0$ lower solution, $\beta = \frac{b}{a}$ upper solution
- **Warning:** superquadratic growth w.r. to the gradient
- v solution, $0 \leq v \leq \frac{b}{a} \Longrightarrow v$ decreasing and concave
- one-sided Bernstein-Nagumo condition $\Longrightarrow$ gradient bound
- Schauder theorem $\Longrightarrow$ solvability

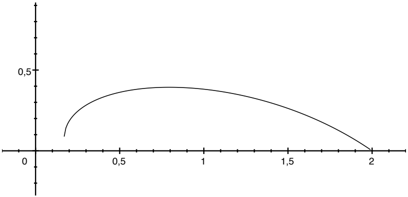
Existence vs non-existence of classical solutions

Remark In spherical shells the previous argument does not work anymore. Now derivatives of solutions change sign: the right hand side of the equation exhibits a genuine (i.e., two-sided) cubic growth w.r. to the gradient

$$-v'' = -av(1 + v'^2)^{3/2} + b(1 + v'^2) + \frac{N-1}{t} v'(1 + v'^2).$$

Gradient's blow up might occur! As numerical simulations show:





Existence vs non-existence of classical solutions

Existence vs non-existence of classical radial solutions in spherical shells

Numerical simulations show:

- the spherical shell $S_{r,R}$ is thin, i.e., $R - r \ll 1$
  $\Longrightarrow$ the solution is classical
- the spherical shell $S_{r,R}$ is thick, i.e., $R - r \gg 1$:
  - $r \gg 1 \quad \Longrightarrow$ the solution is classical
  - $r \ll 1 \quad \Longrightarrow$ the solution is singular (for $b \gg 1$).

Questions

How to get an analytical proof?

How to discuss the case of general domains?

More sophisticated tools of investigation seem to be required.

Variational formulation

The change of variable $v = e^{-bu}$ transforms

$$(Q) \quad \begin{cases} -\mathrm{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = -au + \dfrac{b}{\sqrt{1+|\nabla u|^2}} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

into

$$(P) \quad \begin{cases} -\mathrm{div}\left(\dfrac{\nabla v}{\sqrt{v^2+b^{-2}|\nabla v|^2}}\right) = -a\ln v - \dfrac{b^2 v}{\sqrt{v^2+b^{-2}|\nabla v|^2}} & \text{in } \Omega \\ v = 1 & \text{on } \partial\Omega, \end{cases}$$

whose associated functional is

$$\mathcal{K}(v) = \int_\Omega \sqrt{v^2 + b^{-2}|\nabla v|^2} + \frac{a}{b^2}\int_\Omega v^+(\ln v^+ - 1).$$

Variational formulation

- Due to the linear growth w.r.t. to the gradient of $\int_\Omega \sqrt{v^2 + b^{-2}|\nabla v|^2}$, the natural domain of the functional $\mathcal{K}$ would be $W^{1,1}(\Omega)$, yet $W^{1,1}(\Omega)$ is not a favorable framework where to settle variational methods. Indeed, the appropriate space is $BV(\Omega)$.
- Relaxation of $\mathcal{K}$ from $W^{1,1}(\Omega)$ to $BV(\Omega)$:

$$\mathcal{I}(v) = \int_\Omega \sqrt{v^2 + b^{-2}|Dv|^2} + \frac{a}{b^2}\int_\Omega v^+(\ln v^+ - 1) + \frac{1}{b}\int_{\partial\Omega} |v-1|,$$

where

$$\int_\Omega \sqrt{v^2 + b^{-2}|Dv|^2} = \inf\left\{\liminf_{n\to+\infty} \int_\Omega \sqrt{w_n^2 + b^{-2}|\nabla w_n|^2} \quad | \quad w_n \in W^{1,1}(\Omega), w_n \to v \text{ in } L^1(\Omega)\right\}.$$

- The functional $\int_\Omega \sqrt{v^2 + b^{-2}|Dv|^2} + \frac{1}{b}\int_{\partial\Omega} |v-1|$ satisfies suitable semicontinuity, approximation and lattice properties.

Minimization and regularity

Global minimization

The functional $\mathcal{I}$ has a unique global minimizer $v \in BV(\Omega)$, with $e^{-\frac{b^2}{a}} \leq v \leq 1$.

Sketch of proof (standard)

- $\mathcal{I}$ is bounded from below
- $\mathcal{I}$ is lower semicontinuous w.r. to the L^1-convergence in $BV(\Omega)$

  $\Longrightarrow \exists$ a minimizer, positive and bounded
- $\mathcal{I}$ is strictly convex on the positive cone

  $\Longrightarrow$ the minimizer is unique.

Minimization and regularity

Interior regularity

$v \in C^{\infty}(\Omega) \cap W^{1,1}(\Omega)$

Sketch of proof (quite technical)

- construction of a sequence of local approximating problems (extending some ideas from Gerhardt, 1974)
- Serrin's type existence result (proven for anisotropic equations by Marquardt, 2009) for solving the approximating problems
- classical Ladyzhenskaya-Uraltseva gradient estimates
- Schauder estimates.

Existence of generalized solutions

- Set $u = -\frac{1}{b} \ln v \in C^\infty(\Omega) \cap W^{1,1}(\Omega)$.
- u satisfies :
  - $0 \leq u \leq \frac{b}{a}$
  - $\operatorname{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \in L^\infty(\Omega)$
  - $-\operatorname{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = -au + \dfrac{b}{\sqrt{1+|\nabla u|^2}}$ a.e. in Ω
  - for $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega$
    - either $u(x) = 0$
    - or else $u(x) > 0$ and $\left[\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}, \nu\right] = -1$
  - u minimizes in $W^{1,1}(\Omega) \cap L^\infty(\Omega)$ the functional

$$\int_\Omega e^{-bz}\sqrt{1+|\nabla z|^2} - \frac{a}{b}\int_\Omega e^{-bz}\left(z + \frac{1}{b}\right) + \frac{1}{b}\int_{\partial\Omega} |z-1|$$

  - u is stable.

Existence of generalized solutions

Conclusion: existence and uniqueness of a generalized solution

$\forall N \geq 2 \; \forall a > 0 \; \forall b > 0$ problem (P) has a unique generalized solution.

Questions

At which points of the boundary is the Dirichlet condition attained?

When are the generalized solutions classical?

A detailed study of the boundary behaviour of the generalized solutions is in order.

Boundary behaviour

(Upper and lower solutions come back into the scene)

Generalized upper and lower solutions

▷ β is generalized upper solution of (P) if

- $\beta \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ and $\operatorname{div}\left(\dfrac{\nabla\beta}{\sqrt{1+|\nabla\beta|^2}}\right) \in L^N(\Omega)$
- β satisfies
  - a.e. in Ω
  $$-\operatorname{div}\left(\frac{\nabla\beta}{\sqrt{1+|\nabla\beta|^2}}\right) \geq a\beta + \frac{b}{\sqrt{1+|\nabla\beta|^2}}$$
  - $\mathcal{H}^{N-1}$-a.e. on $\partial\Omega$
    - either $\beta(x) \geq 0$
    - or else $\beta(x) < 0$ and $\left[\dfrac{\nabla\beta}{\sqrt{1+|\nabla\beta|^2}}, \nu\right](x) = 1.$

▷ A generalized lower solution α is defined similarly, by reversing signs.

Boundary behaviour

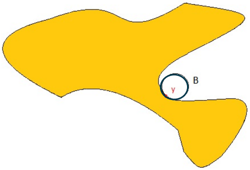
Comparison principle and localization

α generalized lower solution, β generalized upper solution, u generalized solution $\implies$ $\alpha \leq u \leq \beta$

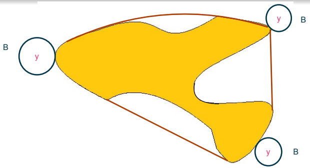
▷ Constructing suitable lower and upper solutions (barriers) may force the solution to attain the boundary conditions

A geometric condition

Ω satisfies at $y \in \partial\Omega$ an
exterior sphere condition of radius r
if $\exists$ an open ball B of radius $r > 0$
s.t. $B \cap \Omega = \emptyset$ and $y \in \partial B$



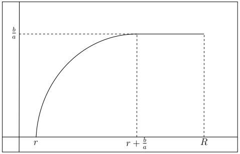
Remark (obvious)
All points of $\partial\Omega$ belonging to the boundary of the convex hull of Ω satisfy an exterior sphere condition of arbitrary radius.



Boundary behaviour

Construction of an upper solution

Ω satisfies at $y \in \partial\Omega$ an exterior sphere condition with radius $r \geq (N-1)\frac{b}{a}$
$\implies$ $\exists\beta$ positive radial upper solution s.t. $\beta(y) = 0$, defined on a spherical shell including Ω and having inner radius r



Conclusion: continuity at the boundary

Ω satisfies at $y \in \partial\Omega$ an exterior sphere condition with radius $r \geq (N-1)\frac{b}{a}$
$\implies$ $\exists\beta$ upper solution s.t. $\beta(y) = 0$ $\implies$ $0 \leq u \leq \beta$
$\implies$ u is continuous at $y \in \partial\Omega$ and $u(y) = 0$

Remark
For any Lipschitz domain Ω and any $a, b > 0$, the subset of $\partial\Omega$ where u attains the boundary conditions is non-empty.

Classical versus singular solutions

Here: $\Omega \in C^{2,\alpha}$.

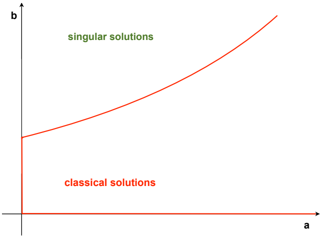
Solvability patterns: revisited

$\forall N \geq 2 \quad \forall a > 0$:

- either $\forall b > 0$ $\exists!$ generalized solution u which is classical
- or else $\exists\, \hat{b} = \hat{b}(a) > 0$ s.t.
  - $0 < b \leq \hat{b} \quad \Longrightarrow \quad \exists!$ generalized solution u which is classical
  - $b > \hat{b} \quad \Longrightarrow \quad \exists!$ generalized solution u which is singular

Additional information:

- $(a, b) \mapsto u(a, b)$ continuous in $L^\infty(\Omega)$
- $\forall a > 0$, $b \mapsto u(a, b)$ increasing
- $\forall b > 0$, $a \mapsto u(a, b)$ decreasing



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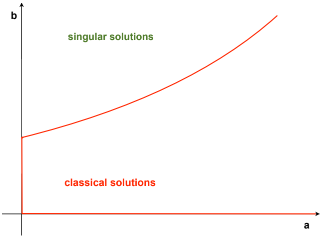
  - $0 < b \leq \hat{b} \quad \Longrightarrow \quad \exists!$ generalized solution u which is classical
  - $b > \hat{b} \quad \Longrightarrow \quad \exists!$ generalized solution u which is singular

Question:
do singular solutions really exist?



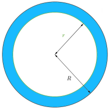
The case of spherical shells: conclusion

Existence of classical solutions on thin spherical shells

$\forall N \geq 2 \quad \forall a > 0 \quad \forall b > 0 \ \forall r > 0 \quad \exists R_* > r \quad \text{s.t.} \quad R \in]r, R_*[$

$\Longrightarrow \quad \exists!$ generalized solution which is classical,

with $u(x) = v(|x|)$, $v \in C^2([r, R])$, $v(r) = 0$, $v(R) = 0$



Sketch of proof

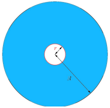
Construct a radial upper solution satisfying the Dirichlet boundary conditions

Existence of singular solutions on thick spherical shells

$\forall N \geq 2 \quad \forall a > 0 \quad \forall r > 0 \quad \exists R^* > 0 \quad \exists b^* > 0 \quad \text{s.t.} \quad R > R^* \quad b > b^*$

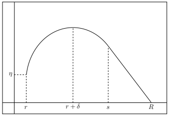
$\Longrightarrow \quad \exists!$ generalized solution which is singular, with

$u(x) = v(|x|)$, $v \in C^2(]r, R])$, $v(r) > 0$, $v'(r) = +\infty$, $v(R) = 0$



Sketch of proof

Construct a singular radial lower solution



Thank you for so much your kind attention!