Fuzzy Stochastic Differential Equations: A Tool for Stochastic Systems with Imprecise Values

Marek T. Malinowski

Institute of Mathematics, Cracow University of Technology,
Kraków, Poland

Differential Equations and Applications,

Brno, Czech Republic, September 4–7, 2017

Stochastic differential equations

$(\Omega, \mathcal{A}, P)$ - a complete probability space.
$\{\mathcal{A}_t\}_{t\in[0,T]}$ - a filtration satisfying usual hypotheses.

$$dx(t) = f(t, x(t))dt + g(t, x(t))dB(t), \qquad x(0) = x_0$$

$$x(t) = x_0 + \int_0^t f(s, x(s))ds + \int_0^t g(s, x(s))dB(s), \qquad t \in [0, T],$$

$f, g\colon [0, T] \times \mathbf{R}^d \to \mathbf{R}^d$,
x_0 is an $\mathcal{A}_0$-measurable random variable,
$\{B(t), t \in [0, T]\}$ - Brownian motion.

Solution: a stochastic process $x\colon [0, T] \times \Omega \to \mathbf{R}^d$.

Stochastic differential equations

Applications: biology, financial mathematics, control theory, physics, economics, mechanical and electrical engineering.

Monographs (among others):

1. L. Arnold, Stochastic Differential Equations: Theory and Applications, John Willey & Sons, New York 1974.
2. I.I. Gihman, A.V. Skorohod, Stochastic Differential Equations, Springer, Berlin 1972.
3. N. Ikeda, S. Watanabe, Stochastic Differential Equations and Diffusion Processes, Kodansha, Tokyo 1981.
4. H. Kunita, Stochastic Flows and Stochastic Differential Equations, Cambridge Univ. Press, Cambridge 1990.
5. X. Mao, Stochastic Differential Equations and Applications, Horwood Publishing, Chichester 1997.

Fuzzy sets

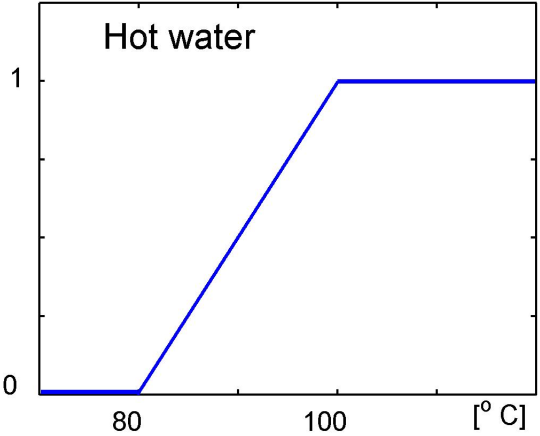
Lotfi A. Zadeh 1965:

A **fuzzy set** (class) u in $\mathbf{R}^d$ is characterized by a membership function $u\colon \mathbf{R}^d \to [0,1]$ which associates with each point in $\mathbf{R}^d$ a real number in the interval $[0,1]$.

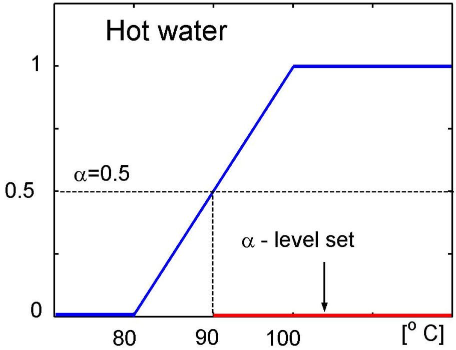
The value $u(x)$ represents the grade of membership of x in u.

Fuzzy sets

Example: the class "Hot water"



Fuzzy sets



$[u]_\alpha := \{x \in \mathbf{R} : u(x) \geq \alpha\}$ – α-level of the fuzzy set u, $\alpha \in (0,1]$,

$[u]_0 := \mathrm{cl}\{x \in \mathbf{R} : u(x) > 0\}$ – support of u.

Fuzzy sets

Notation

$$\mathcal{F}(\mathbf{R}^d) := \Big\{ u : \mathbf{R}^d \to [0,1] \mid [u]_\alpha \text{ is a nonempty, convex and compact subset of } \mathbf{R}^d \text{ for } \alpha \in [0,1] \Big\}.$$

Two sources of uncertainties

Fuzzy Stochastic Differential Equations (FSDEs)

$$dx(t) = f(t, x(t))dt + g(t, x(t))dB(t), \qquad x(0) = x_0$$

$f, g\colon [0,T] \times \mathcal{F}(\mathbf{R}^d) \to \mathcal{F}(\mathbf{R}^d)$ - **fuzzy coefficients**,

$x_0\colon \Omega \to \mathcal{F}(\mathbf{R}^d)$ is an $\mathcal{A}_0$-measurable **fuzzy random variable**,

$\{B(t), t \in [0,T]\}$ - real-valued Brownian motion.

Solution: a **fuzzy stochastic process** $x\colon [0,T] \times \Omega \to \mathcal{F}(\mathbf{R}^d)$.

Integral form of FSDEs

$$x(t) = x_0 + \int_0^t f(s, x(s))ds + \int_0^t g(s, x(s))dB(s), \quad t \in [0, T],$$

- addition of fuzzy sets,
- fuzzy stochastic Lebesgue integral,
- fuzzy stochastic Itô integral.

Addition of fuzzy sets $u, v \in \mathcal{F}(\mathbf{R}^d)$**:**

$u + v$ is a fuzzy set in $\mathcal{F}(\mathbf{R}^d)$ such that for every $\alpha \in [0, 1]$

$$[u + v]_\alpha = [u]_\alpha + [v]_\alpha \qquad \leftarrow \quad \text{Minkowski's sum of sets.}$$

Fuzzy stochastic integrals

$a\colon [0,T] \times \Omega \to \mathcal{F}(\mathbf{R}^d)$ - measurable, adapted and integrally bounded fuzzy stochastic process

$$\int_0^t a(s,\omega)ds \text{ - } \textbf{fuzzy stochastic Lebesgue integral},$$

$$\left[\int_0^t a(s,\omega)ds\right]_\alpha = \int_0^t [a(s,\omega)]_\alpha ds \;\leftarrow \text{Aumann's integral.}$$

Theorem 1 (MTM 2009, 2012).

The mapping $[0,T] \times \Omega \ni (t,\omega) \mapsto \int_0^t a(s,\omega)ds \in \mathcal{F}(\mathbf{R}^d)$ is a measurable, adapted and integrally bounded fuzzy stochastic process. Its trajectories are continuous.

Fuzzy stochastic integrals

b - fuzzy stochastic process,
B - real-valued Brownian motion,

$$\int_0^t b(s)dB(s)$$ - **fuzzy stochastic Itô integral**.

Theorem 2 (Ogura & Zhang 2010).

It is not possible to define a fuzzy stochastic Itô integral $\left(\int_0^t b(s)dB(s)\right)(\omega)$ in such a way that it is a fuzzy (and non-crisp) stochastic process.

Fuzzy stochastic differential equations

$$x(t) = \underbrace{x_0}_{\text{fuzzy-valued}} + \underbrace{\int_0^t f(s, x(s))ds}_{\text{fuzzy-valued}} + \underbrace{\int_0^t g(s, x(s))dB(s)}_{\text{crisp}},$$

$f\colon [0,T] \times \Omega \times \mathcal{F}(\mathbf{R}^d) \to \mathcal{F}(\mathbf{R}^d)$,
$g\colon [0,T] \times \Omega \times \mathcal{F}(\mathbf{R}^d) \to \mathbf{R}^d$,
$x_0\colon \Omega \to \mathcal{F}(\mathbf{R}^d)$ is a fuzzy random variable,
B- real-valued Brownian motion.

Fuzzy stochastic differential equations

$$dx(t) = f(t, x(t))dt + \langle g(t, x(t))dB(t) \rangle, \quad x(0) = x_0 \tag{1}$$

$$x(t) = x_0 + \int_0^t f(s, x(s))ds + \left\langle \int_0^t g(s, x(s))dB(s) \right\rangle, \quad t \in [0, T],$$

Definition 3.

By a **solution to Eq. (1)** we mean a measurable, adapted and integrally bounded fuzzy stochastic process $x\colon [0, T] \times \Omega \to \mathcal{F}(\mathbf{R}^d)$ with continuous trajectories and such that P-a.e.

$$x(t) = x_0 + \int_0^t f(s, x(s))ds + \left\langle \int_0^t g(s, x(s))dB(s) \right\rangle, \quad t \in [0, T].$$

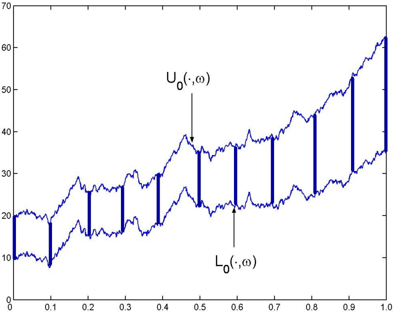
The **solution** x **is unique** if P-a.e. $x(t) = y(t), \ t \in [0, T]$, where $y\colon [0, T] \times \Omega \to \mathcal{F}(\mathbf{R}^d)$ is any other solution to Eq. (1).

Fuzzy stochastic differential equations

Example: $x(t) = x_0 + \int_0^t 2sx(s)ds + \left\langle \int_0^t 4dB(s) \right\rangle$.

Solution: $x(t) = e^{t^2} \cdot x_0 + \left\langle 4e^{t^2} \int_0^t e^{-s^2} dB(s) \right\rangle$.

For a simulation: $[x_0(\omega)]_0 = [10, 20]$, $[x(t,\omega)]_0 = [L_0(t,\omega), U_0(t,\omega)]$.



Fuzzy stochastic differential equations

Notation

$$\mathrm{Fuzz}(u) := \mathrm{diam}([u]_0) \text{ for } u \in \mathcal{F}(\mathbf{R}^d).$$

Proposition 4 (MTM 2013).

Assume that $x\colon [0,T] \times \Omega \to \mathcal{F}(\mathbf{R}^d)$ is a solution to Eq. (1). Then P-a.a. the functions $t \mapsto \mathrm{Fuzz}(x(t,\omega))$ are nondecreasing.

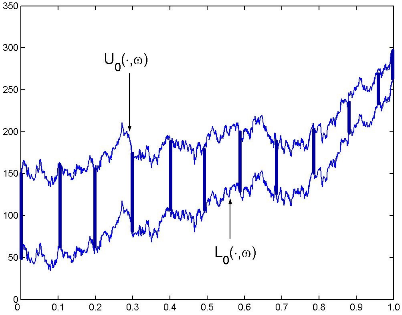
Fuzzy stochastic differential equations

Example: $x(t) + \int_0^t (-2)sx(s)ds + \left\langle \int_0^t (-4)dB(s) \right\rangle = x_0.$

Solution:

$x(t) = \left[\cosh(t^2) \cdot x_0 \ominus \left(-\sinh(t^2) \cdot x_0\right)\right] + \left\langle 4e^{t^2} \int_0^t e^{-s^2} dB(s) \right\rangle.$

For a simulation: $[x_0(\omega)]_0 = [50, 150]$, $[x(t,\omega)]_0 = [L_0(t,\omega), U_0(t,\omega)]$.



Fuzzy stochastic differential equations

Consider

$$x(t)+\int_0^t (-1)f(s,x(s))ds+\Big\langle \int_0^t (-1)g(s,x(s))dB(s)\Big\rangle = x_0. \qquad (2)$$

Proposition 5 (MTM 2014).

Assume that $x\colon [0,T]\times\Omega\to\mathcal{F}(\mathbf{R^d})$ is a solution to Eq. (2). Then P-a.a. the functions $t\mapsto \mathrm{Fuzz}(x(t,\omega))$ are nonincreasing.

Bipartite fuzzy stochastic differential equations

Bipartite fuzzy stochastic differential equations

$$\begin{aligned} x(t) &+ \int_0^t (-1) f_1(s, x(s)) ds + \Big\langle \int_0^t (-1) g_1(s, x(s)) dB_1(s) \Big\rangle \\ &= x_0 + \int_0^t f_2(s, x(s)) ds + \Big\langle \int_0^t g_2(s, x(s)) dB_2(s) \Big\rangle. \end{aligned}$$

$$\begin{aligned} x(t) &+ \int_0^t (-1) f_1(s, x(s)) ds \\ &= x_0 + \int_0^t f_2(s, x(s)) ds + \Big\langle \int_0^t g(s, x(s)) dB(s) \Big\rangle. \qquad (3) \end{aligned}$$

$g = (g_1, g_2) \qquad B = (B_1, B_2)'.$

Bipartite fuzzy stochastic differential equations

Assumptions:

(a1) $f_k(\cdot,\cdot,\cdot)\colon [0,T]\times\Omega\times\mathcal{F}(\mathbf{R}^d)\to\mathcal{F}(\mathbf{R}^d)$, $k=1,2$, $g(\cdot,\cdot,\cdot)\colon [0,T]\times\Omega\times\mathcal{F}(\mathbf{R}^d)\to\mathbf{R}^d$ are jointly measurable mappings,

(a2) there exists a constant $L>0$ such that $\forall (t,\omega)\in[0,T]\times\Omega \quad \forall u,v\in\mathcal{F}(\mathbf{R}^d)$

$$\max\Big\{D\big(f_k(t,\omega,u),f_k(t,\omega,v)\big),\|g(t,\omega,u)-g(t,\omega,v)\|\Big\}\leqslant LD(u,v),$$

(a3) there exists a constant $C>0$ such that $\forall (t,\omega)\in[0,T]\times\Omega$

$$\max\Big\{D\Big(f_k(t,\omega,\langle 0\rangle),\langle 0\rangle\Big),\|g(t,\omega,\langle 0\rangle)\|\Big\}\leqslant C.$$

Bipartite fuzzy stochastic differential equations

Theorem 6 (MTM 2016).

Let assumptions (a1)-(a3) be satisfied. Suppose that the sequence $\{x_n\}_{n=0}^{\infty}$ of the fuzzy stochastic processes $x_n\colon [0,\tilde{T}]\times\Omega\to\mathcal{F}(\mathbb{R})$, where $\tilde{T}\leqslant T$, described as: $x_0(t)=x_0$ and for $n=1,2,\ldots$

$$\begin{aligned} x_n(t) &= \left(\left[x_0+\int_0^t f_2(s,x_{n-1}(s))ds\right]\ominus\int_0^t(-1)f_1(s,x_{n-1}(s))ds\right)\\ &\quad+\left\langle\int_0^t g(s,x_{n-1}(s))dB(s)\right\rangle \end{aligned}$$

is well defined. Then Eq. (3) possesses a unique local solution.

Bipartite fuzzy stochastic differential equations

Assumptions:

(b1) $f_k(\cdot,\cdot,\cdot)\colon [0,T]\times\Omega\times\mathcal{F}_c(\mathbf{R}^d)\to\mathcal{F}_c(\mathbf{R}^d)$, $k=1,2$,
$g(\cdot,\cdot,\cdot)\colon [0,T]\times\Omega\times\mathcal{F}_c(\mathbf{R}^d)\to\mathbf{R}^d$ are jointly measurable mappings,

(b2) $\forall (t,\omega)\in[0,T]\times\Omega \quad \forall u,v\in\mathcal{F}_c(\mathbf{R}^d)$

$$\max\Big\{D^2\big(f_k(t,\omega,u),f_k(t,\omega,v)\big),\|g(t,\omega,u)-g(t,\omega,v)\|^2\Big\}\leqslant\xi(D^2(u,v)),$$

where $\xi\colon \mathbf{R}_+\to\mathbf{R}_+$ is continuous, concave, nondecreasing, $\xi(0)=0$, $\xi(a)>0$ for $a>0$, and $\int_{0+}\frac{da}{\xi(a)}=\infty$,

(b3) there exists a constant $C>0$ such that $\forall (t,\omega)\in[0,T]\times\Omega$

$$\max\left\{D^2\big(f_k(t,\omega,\langle 0\rangle),\langle 0\rangle\big),\|g(t,\omega,\langle 0\rangle)\|^2\right\}\leqslant C.$$

Bipartite fuzzy stochastic differential equations

Theorem 7 (MTM 2017).

Let assumptions (b1)-(b3) be satisfied. Suppose that there exists $\tilde{T} \in (0, T]$ such that

$$\int_{\tau}^{t} f_2(s, x(s))ds \ominus \int_{\tau}^{t} f_1(s, x(s))ds$$

is well defined for all $\tau, t \in [0, \tilde{T}]$ and for every fuzzy stochastic process x. Then Eq. (3) possesses a unique local solution.

References

1. E.J. Jung, J.H. Kim (2003): On set-valued stochastic integrals. **Stoch. Anal. Appl.** 21, 401–418.
2. M.T. Malinowski (2009): On random fuzzy differential equations. **Fuzzy Sets Syst.** 160, 3152–3165.
3. M.T. Malinowski (2010): Existence theorems for solutions to random fuzzy differential equations. **Nonlinear Anal. Theory, Meth. & Appl.** 73, 1515–1532.
4. M.T. Malinowski (2012): Strong solutions to stochastic fuzzy differential equations of Itô type. **Math. Comput. Modelling** 55, 918–928.
5. M.T. Malinowski (2012): Itô type stochastic fuzzy differential equations with delay. **Syst. Control Lett.** 61, 692–701.
6. M.T. Malinowski (2013): Some properties of strong solutions to stochastic fuzzy differential equations. **Inform. Sci.** 252, 62–80.

References

7. M.T. Malinowski (2015): Set-valued and fuzzy stochastic differential equations in M-type 2 Banach spaces, **Tôhoku Math. J.** 67, 349–381.
8. M.T. Malinowski, R.P. Agarwal (2015): On solutions to set-valued and fuzzy stochastic differential equations. **J. Franklin Inst.**, 352, 3014–3043.
9. M.T. Malinowski (2015): Fuzzy and set-valued stochastic differential equations with local Lipschitz condition. **IEEE Trans. Fuzzy Syst.** 23, 1891–1898.
10. M.T. Malinowski (2016): Stochastic fuzzy differential equations of a nonincreasing type. **Commun. Nonlinear Sci. Numer. Simulat.** 33, 99-117.
11. M.T. Malinowski (2016): Bipartite fuzzy stochastic differential equations with global Lipschitz condition. **Math. Prob. Eng.** vol. 2016, 13 pages.
12. Y. Ogura, J. Zhang (2010): On stochastic differential equations with fuzzy set coefficients. In: Soft Methods for Handling Variability and Imprecision, ASC 48, (D. Dubois et al., Eds.), Springer, Berlin.

Thank you for your attention!