Protter-Morawetz problem for (3+1)-D equations of Keldysh type

Tsvetan Hristov

Faculty of Mathematics and Informatics, Sofia University, Bulgaria
tsvetan@fmi.uni-sofia.bg

Conference "Differential equations and applications"
Brno, Czech Republic, September 4-7 2017

The talk is based on several joint works with:

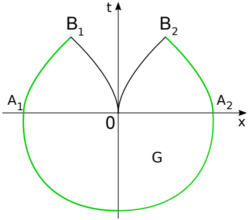
- Nedyu Popivanov
  - Institute of Information and Communication Technologies, Bulgarian Academy of Sciences, Sofia, Bulgaria
  - Faculty of Mathematics and Informatics, Sofia University, Bulgaria,

  e-mail: nedyu@fmi.uni-sofia.bg
- Manfred Schneider
  Faculty of Mathematics, Karlsruhe Institute of Technology, Germany,
  e-mail: manfred.schneider@kit.edu
- Aleksey Nikolov
  Faculty of Applied Mathematics and Informatics, Technical University of Sofia, Bulgaria,
  e-mail: alekseyjnikolov@gmail.com

1. Introduction and brief historical overview

Transonic potential flows around airfoils are modeled in the hodograph plane by the Guderley-Morawetz problem for the Chaplygin equation (with $tK(t) > 0$ for $t \neq 0$):

$$\left| \begin{array}{l} K(t)u_{xx} - u_{tt} = f(x,t) \quad \text{in} \quad G, \\ u = 0 \quad \text{on} \quad \sigma \cup A_1B_1 \cup A_2B_2. \end{array} \right.$$



Morawetz (1958): Existence of weak solutions and uniqueness of a strong solution in weighted Sobolev spaces.

Lax and Phillips (1960): The weak solutions are strong. Regularity.

- L. Bers, Mathematical aspects of subsonic and transonic gas dynamics, John Wiley and Sons, Ltd, 1958.
- C. Morawetz, Mixed equations and transonic flow, J. Hyperbolic Differ. Equ., Vol. 1, No. 1, 1-26 (2004).
- A. Kuz'min, Boundary Value Problems for Transonic Flow, John Wiley and Sons, Ltd, 2002.

1.1. The multidimensional Protter-Morawetz problem

Find a solution to equation ($x = (x_1, \ldots, x_N),\ N \geq 2$)

$$K(t)\sum_{j=1}^{N} u_{x_j x_j} - u_{tt} = f(x,t) \quad \text{in} \quad \Omega,$$

that satisfies the boundary condition: $u = 0$ on $\Gamma \cup S_1$.

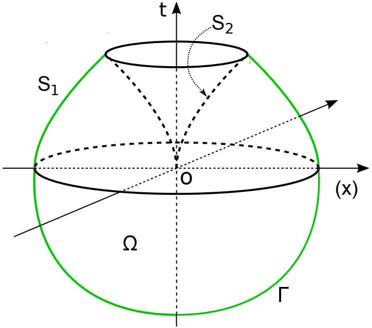


Figure: Protter-Morawetz domain.

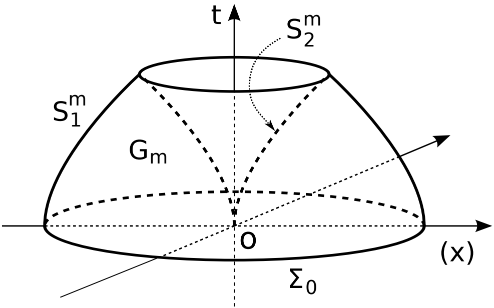
- M. Protter proposed an interesting multidimensional generalization of 2-D Guderley-Morawetz problem in the 1950s.
- Aziz and Schneider (1979) obtained an uniqueness result for this problem.
- The Protetr-Morawetz problem have been studied by many authors, but a general understanding of the situation is still not at hand. Even now there is not a single example of a nontrivial solution to the multidimensional problem, neither a general existence result is known.
- Many difficulties and differences in comparison with the planar problems can be illustrated as well by the related problems in the hyperbolic part of the domain.

1.2. Protter-Morawetz problem for weakly hyperbolic quations of Tricomi type

Consider the equation of Tricomi type ($m \in \mathbf{R},\ m \geq 0$)

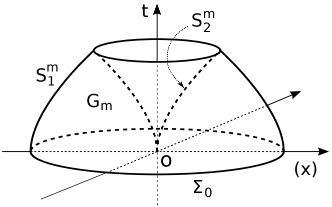
$$l_m[u] := t^m \sum_{j=1}^{N} u_{x_j x_j} - u_{tt} = f(x,t) \qquad (1)$$

in the domain G_m, bounded by the ball $\Sigma_0 := \{t = 0,\ |x| < 1\}$ and by two characteristics surfaces of (1) S_1^m, and S_2^m.



Thew following multidimensional analogues of the plane Darboux problem were proposed by M. Protter:

$$P1: \left| \begin{array}{l} l_m[u] = f(x,t) \;\text{ in }\; G_m, \\ u|_{\Sigma_0 \cup S_1^m} = 0 \end{array} \right.$$



$$P2: \left| \begin{array}{l} l_m[u] = f(x,t) \;\text{ in }\; G_m, \\ u|_{S_1^m} = 0, u_t|_{\Sigma_0} = 0 \end{array} \right.$$

- Garabedian (1960) proved the uniqueness of a classical solution to problem $P1$ for the wave equation $(m = 0)$ in $\mathbb{R}^4$.
- Popivanov, Schneider (1993) and Khe Kan Cher (1998) showed that both problems $P1$ and $P2$ are not well-posed in the frame of classical solvability, since they have infinite-dimensional co-kernels
- Popivanov, Schneider (1993) suggested to study the Protter problems in the frame of generalized solutions with possible big singularities. Today it is well known that the Protter problems have singular generalized solutions, even for smooth right-hand sides.
- Aldashev (2005) studied some of the Protter problems and different their generalizations (including some applications in the industrial explosion process).
- Different aspects of Protter problems and several their generalizations are studied also by J. Choi, J. Park, Ar. Bazarbekov, Ak. Bazarbekov, M. Grammatikopoulos, R. Scherer, A. Tesdall, D. Lupo, K. Payne.

1.3. Equations of Keldysh type

The 2-D Keldysh type equations ($m \in \mathbf{R}$, $m > 0$)

$$u_{xx} + t^m u_{tt} + au_x + bu_t + cu = 0 \qquad (2)$$

are also known to play an important role in fluid mechanics.

- Keldysh (1951) showed that for degenerating elliptic equation (2) the formulation of the Dirichlet problem may depend on the lower order terms.
- Fichera (1956) generalized Keldysh's results for multidimensional linear second order equations with non-negative characteristic form and now BVPs for them are well understood in the sense that boundary conditions should not be imposed on the whole boundary.
- A summary of Fichera's theory can be found in

E. Radkevich, Equations with nonnegative characteristic form, I and II, J. Math. Sci., 2009.

- Otway (2010) and Lupo, Monticelli, Payne (2015) gave a statement of some 2-D BVPs for elliptic-hyperbolic Keldysh-type equations with specific applications in plasma physics, including a model for analyzing the possible heating in axisymmetric cold plasmas.
- Čanić and Keyfitz (1996) studied some problems for a nonlinear degenerate elliptic equation, whose solutions behave like those of a Keldysh-type equation. Such an equation arises in the modeling of a weak shock reflection at a wedge.
- Keyfitz (2006) examined whether the Fichera's classification could be extended to quasilinear equations and mentioned that contrasting behavior of the characteristics of the Tricomi and Keldysh equation may have implications, unexplored yet, for the solution of some free boundary problems arising in the fluid dynamics models.

2. Protter-Morawetz problem for weakly hyperbolic quations of Keldysh type

For $m \in \mathbb{R}$, $0 < m < 2$ we consider the equation

$$L_m[u] \equiv u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3} - (t^m u_t)_t = f(x,t) \qquad (3)$$

in the domain

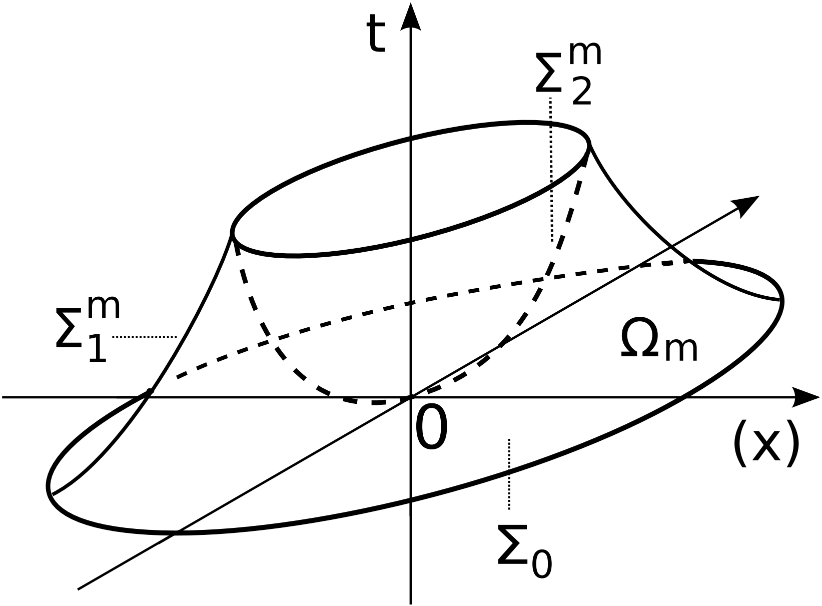
$$\Omega_m := \left\{ (x,t) : \quad 0 < t < t_0, \quad \frac{2}{2-m} t^{\frac{2-m}{2}} < |x| < 1 - \frac{2}{2-m} t^{\frac{2-m}{2}} \right\},$$

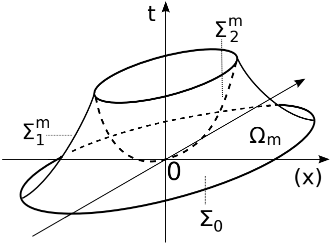
bounded by the ball $\Sigma_0 := \{(x,t) : \quad t = 0, \quad |x| < 1\}$ and by two characteristic surfaces of equation (3)

$$\Sigma_1^m := \left\{ (x,t) : 0 < t < t_0, \ |x| = 1 - \frac{2}{2-m} t^{\frac{2-m}{2}} \right\},$$

$$\Sigma_2^m := \left\{ (x,t) : 0 < t < t_0, \ |x| = \frac{2}{2-m} t^{\frac{2-m}{2}} \right\},$$

where $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ and $t_0 = \left(\frac{2-m}{4}\right)^{\frac{2}{2-m}}$.


t
Σ_2^m
Σ_1^m
Ω_m
0
(x)
Σ_0



Problem $\mathbf{PK}$. Find a solution to equation $L_m[u] = f$ in Ω_m which satisfies the boundary conditions

$$u|_{\Sigma_1^m} = 0, \qquad t^m u_t \to 0 \text{ as } t \to +0.$$

Problem $\mathbf{PK}^*$. Find a solution to the self-adjoint equation $L_m[u] = f$ in Ω_m which satisfies the boundary conditions

$$u|_{\Sigma_2^m} = 0, \qquad t^m u_t \to 0 \text{ as } t \to +0.$$

Paper

T. Hristov, N. Popivanov, and M. Schneider, On the uniqueness of generalized and quasi- regular solutions for equations of mixed type in R^3, Sib. Adv. Math., Vol. 21, No. 4, 262-273 (2011):
Using the exact Hardy-Sobolev inequality, the uniqueness of a quasi-regular solution to problem PK.

Paper

N. Popivanov, T. Hristov, A. Nikolov, M. Schneider, On the existence and uniqueness of a generalized solution of the Protter problem for (3+1) -D Keldysh-type equations, Boundary Value Problems, Vol. 2017, Paper No. 26, 30 p. (2017)

Paper

N. Popivanov, T. Hristov, A. Nikolov, M. Schneider, Singular solutions to a (3+1)-D Protter-Morawetz problem for Keldysh-type equations, Advances in Mathematical Physics, 24 p. (2017) [in print]

2.1. Solutions to the homogeneous adjoint problem PK^*.

Define the functions

$$E^n_{k,m}(|x|,t) := \sum_{i=0}^{k} A^k_i |x|^{-n+2i-1} \left(|x|^2 - \frac{4}{(2-m)^2} t^{2-m} \right)^{n-k-i-\frac{m}{2(2-m)}},$$

where the coefficients are

$$A^k_i := (-1)^i \frac{(k-i+1)_i (n-k-i+(4-3m)/(4-2m))_i}{i!(n+1/2-i)_i}$$

with $(a)_0 = 1$ and $(a)_i = a(a+1)\dots(a+i-1)$.

Further, let us denote by $Y^s_n(x)$, $n \in \mathbb{N} \cup \{0\}$, $s = 1, 2, \dots, 2n+1$ the three-dimensional spherical functions. They are usually defined on the unit sphere $S^2 := \{x \in \mathbb{R}^3) : |x| = 1\}$, but for convenience of our discussions we extend them out of S^2 radially, keeping the same notation for the extended functions:

$$Y^s_n(x) := Y^s_n(x/|x|), \quad x \in \mathbb{R}^3 \setminus \{0\}.$$

Lemma 1

For all $m \in \mathbb{R},\ 0 < m < 2,\ k, n \in \mathbb{N} \cup \{0\},\ n \geq N(m,k)$ *and* $s = 1, 2, \ldots, 2n+1,$ *the functions*

$$v_{k,m}^{n,s}(x,t) := \begin{cases} E_{k,m}^{n}(|x|,t)Y_n^s(x), & (x,t) \neq O, \\ 0, & (x,t) = O \end{cases} \tag{4}$$

are classical solutions of the homogeneous problem PK^*.

A necessary condition for the existence of a classical solution of problem PK is the orthogonality in $L_2(\Omega_m)$ of the right-hand side function $f(x,t)$ to all functions $v_{k,m}^{n,s}(x,t)$:

$$\begin{aligned} \mu_{k,m}^{n,s} &:= \left(v_{k,m}^{n,s}, f\right)_{L_2(\Omega_m)} = \left(v_{k,m}^{n,s}, L_m[u]\right)_{L_2(\Omega_m)} \\ &= \left(L_m[v_{k,m}^{n,s}], u\right)_{L_2(\Omega_m)} = 0. \end{aligned}$$

2.2. Generalized solutions to problem PK

Definition 2

We call a function $u(x,t)$ a generalized solution of problem PK in Ω_m, $0 < m < \frac{4}{3}$, for equation $L_m[u] = f$ if:

1. $u, u_{x_j} \in C(\bar{\Omega}_m \setminus O), j = 1,2,3,\ u_t \in C(\bar{\Omega}_m \setminus \bar{\Sigma}_0)$;
2. $u|_{\Sigma_1^m} = 0$;
3. For each $\varepsilon \in (0,1)$ there exists a constant $C(\varepsilon) > 0$, such that
$$|u_t(x,t)| \leq C(\varepsilon) t^{-\frac{3m}{4}} \quad \text{in} \quad \Omega_m \cap \{|x| > \varepsilon\}\,;$$
4. The identity
$$\int_{\Omega_m} \{t^m u_t v_t - u_{x_1} v_{x_1} - u_{x_2} v_{x_2} - u_{x_3} v_{x_3} - fv\} dxdt = 0$$
holds for all
$v \in C^2(\bar{\Omega}_m),\ v|_{\Sigma_2^m} = 0,\ v \equiv 0\ \ \textit{in a neighborhood of}\ \ O.$

2.3. Existence and uniqueness results.

Theorem 3

If $m \in (0, \frac{4}{3})$, *then there exists at most one generalized solution of problem* PK *in* Ω_m.

Theorem 4

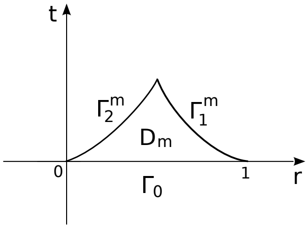
Let $m \in (0, \frac{4}{3})$. *Suppose that the right-hand side function* $f(x,t)$ *is fixed as a "harmonic polynomial" of order* l *with* $l \in \mathbf{N} \cup \{0\}$:

$$f(x,t) = \sum_{n=0}^{l} \sum_{s=1}^{2n+1} f_n^s(|x|,t) Y_n^s(x) \qquad (5)$$

and $f \in C^1(\bar{\Omega}_m)$. *Then there exists an unique generalized solution* $u(x,t)$ *of problem* PK *in* Ω_m *and it has the form*

$$u(x,t) = \sum_{n=0}^{l} \sum_{s=1}^{2n+1} u_n^s(|x|,t) Y_n^s(x). \qquad (6)$$

For the coefficients $u_n^s(r,t)$ which correspond to the right-hand sides $f_n^s(r,t)$ with $r=|x|$ we obtain the $2-D$ Darboux-Goursat problem



$$PK_1 : \left| \begin{array}{l} u_{rr} + \frac{2}{r}u_r - (t^m u_t)_t - \frac{n(n+1)}{r^2}u = f \quad \text{in} \quad D_m, \\ u|_{\Gamma_1^m} = 0; \; t^m u_t \to 0, \text{ as } t \to +0. \end{array} \right.$$

The Riemann-Hadamard function associated to the problem PK_1 :

- It is well unown in the case $n=0$ and $0<m<1$ (S. Gellerstedt, A. Nakhushev, E. Moiseev, M. Smirnov). It involves the hypergeometric function of Gauss ${}_2F_1(a,b,c;\zeta)$.
- We construct it in the case $n \in \mathbb{N} \cup \{0\}$ and $0<m<4/3$, using the Appell series $F_3(a_1,a_2,b_1,b_2,c;x,y)$ and the Horn series $H_2(a_1,a_2,b_1,b_2,c;x,y)$.

Theorem 5

Let $m \in (0, \frac{4}{3})$. *Suppose that the right-hand side function* $f(x,t)$ *is fixed as a "harmonic polynomial"* (5) *of order* l *and* $f \in C^1(\bar{\Omega}_m)$. *Then the unique generalized solution of the Problem* PK *in* Ω_m *has the form* (6) *and satisfies the a priori estimate*

$$|u(x,t)| \leq c \left(\max_{\bar{\Omega}_m} |f| \right) |x|^{-l-1}, \tag{7}$$

with a constant $c > 0$ *independent on* f.

2.4. Exact behavior of the singular generalized solutions.

Theorem 6

Let $m \in (0, \frac{4}{3})$ *and the right-hand side function* $f \in C^1(\bar{\Omega}_m)$ *has the form* (5). *Then the unique generalized solution* $u(x,t)$ *of problem* PK *on the characteristic surface*

$$\Sigma_2^m = \left\{(x,t) : \, 0 < |x| < 1/2, \; t = \tau(|x|) := \left(2^{-1}(2-m)|x|\right)^{\frac{2}{2-m}}\right\}$$

has the following expansion at the point O*:*

$$u(x, \tau(|x|)) = \sum_{p=0}^{l} G_p(x)|x|^{-p-1} + G(x)|x|^{-\frac{m}{2(2-m)}}.$$

(i) The function $G \in C^1(0 < |x| < 1/2)$ *and satisfies the a priori estimate*

$$|G(x)| \leq C \max_{\bar{\Omega}_m} |f|$$

with a constant $C > 0$ *independent of* f*;*

$$u(x,\tau(|x|)) = \sum_{p=0}^{l} G_p(x)|x|^{-p-1} + G(x)|x|^{-\frac{m}{2(2-m)}},$$

(ii) The functions $G_p(x)$ have the following structure

$$G_p(x) = \sum_{k=0}^{[(l-p)/2]} \sum_{s=1}^{2p+4k+1} c_{k,m}^{p+2k,s} \mu_{k,m}^{p+2k,s} Y_{p+2k}^{s}(x), \quad p = 0,1,\ldots,l,$$

where $\mu_{k,m}^{p+2k,s} = \left(v_{k,m}^{p+2k,s}, f\right)_{L_2(\Omega_m)}$, but $c_{k,m}^{p+2k,s} \neq 0$ are constants independent of $f(x,t)$.

Corollary 7

Suppose that at least one of the constants $\mu_{k,m}^{p+2k,s}$ in (22) is different from zero. Then for the corresponding function $G_p(x)$ there exists a vector $\alpha \in \mathbb{R}^3$, $|\alpha| = 1$, such that $G_p(\alpha\varrho) \to c_{p,\alpha} = const \neq 0$ as $\varrho \to +0$. This means that the order of singularity of $u(x,t)$ will be no smaller than $p+1$.

Corollary 8

Let the conditions of Theorem 6 are fulfilled and in addition $f(x,t)$ *satisfies the orthogonality conditions*

$$\int_{\Omega_m} v_{k,m}^{n,s}(x,t) f(x,t)\, dxdt = 0 \tag{8}$$

for all $n = 0, 1, \ldots, l$; $k = 0, 1, \ldots, [\frac{n}{2}]$ *and* $s = 1, 2, \ldots, 2n+1$. *Then the unique generalized solution* $u(x,t)$ *of problem* PK *fulfills on* Σ_2^m *the a priori estimate*

$$|u(x,\tau(|x|))| \leq C \left(\max_{\bar{\Omega}_m} |f| \right) |x|^{-\frac{m}{2(2-m)}},$$

where C *is a positive constant independent of* $f(x,t)$.

Open problems:

1. In the case when the right-hand side function $f(x,t)$ is a "harmonic polynomial" to find additional conditions under which problem PK has a bounded or even classical solution.

2. To study the general case of problem PK when the right-hand side function $f(x,t)$ is a smooth function not only of the form of "harmonic polynomial". What kind of singularity may have the generalized solution in this case?

3. To study problem PK in the more general case when $0 < m < 2$. Let us mention that the presentation of the generalized solution $u(x,t)$, which we are studying, is valid only in the case when $m \in (0, 4/3)$. Find an appropriate technics that works for $4/3 \leq m < 2$.

THANK YOU FOR YOUR ATTENTION!