Non-monotone traveling waves solutions for a monostable reaction-diffusion equations with delay

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Introduction

$$u_t(t,x) = \Delta u(t,x) - u(t,x) + g(u(t-h,x)); \qquad (1)$$

- $h \geq 0$ is the delay,
- $g : \mathbb{R}_+ \to \mathbb{R}_+$ is the birth function.

Introduction

Delayed Diffusive Nicholson's Blowflies Equation [Gurney, Blythe, Nisbet, 1980]

$$u_t(t,x) = \Delta u(t,x) - \delta u(t,x) + pu(t-h,x)e^{-u(t-h,x)} \qquad (2)$$

Introduction

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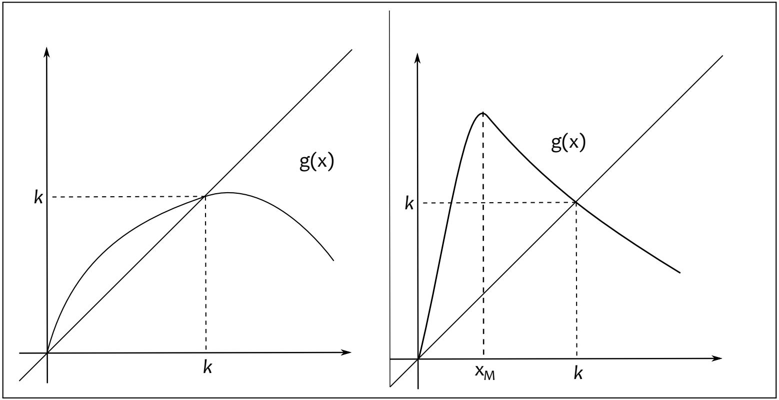
Diffusive Mackey-Glass equation [blood cell production model, 1977]

$$u_t(t,x) = \Delta u(t,x) - u(t,x) + p\frac{u(t-h,x)}{1+(u(t-h,x))^n}; \qquad (3)$$

Introduction

- $g(0) = 0$, $g(\kappa) = \kappa > 0$ is a $C^2(\mathbb{R}_+)$ function.
- $g'(0) > 1$, $g'(\kappa) < 1$
- If g is not strictly increasing between 0 and κ, then it has a unique global maximum in x_M
- $u_0 \equiv 0$, $u_\kappa \equiv \kappa$ are constant solutions to (1)

Introduction

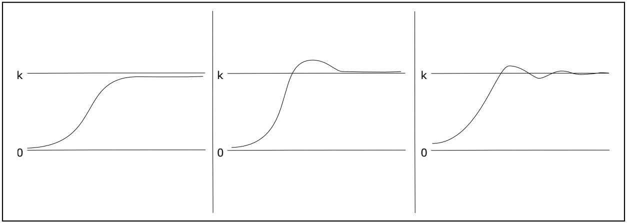


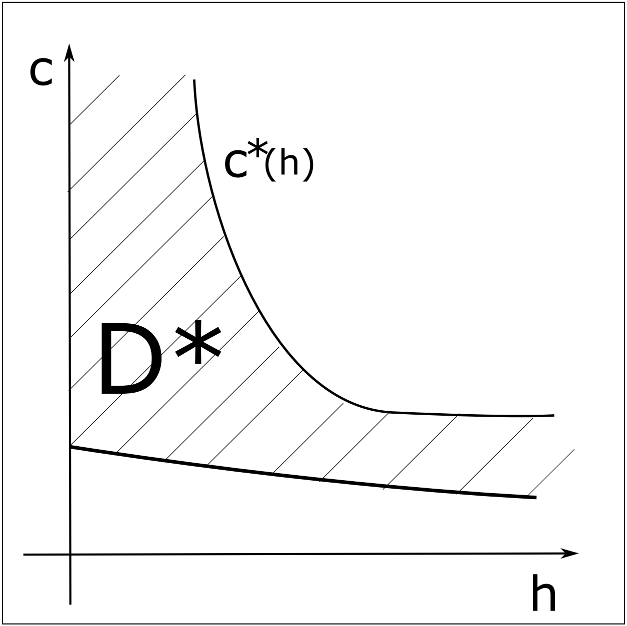
- $u(t,x)$ is a traveling wave solution for (1) if it is positive and $u(t,x) = \phi(x+ct)$, $\phi : \mathbb{R} \to \mathbb{R}$ is a $C^2(\mathbb{R})$, $\phi(-\infty) = 0$, $\phi(+\infty) = \kappa$.
- c is the wave's speed propagation.

$$\phi''(t) - c\phi'(t) - \phi(t) + g(\phi(t - ch)) = 0, \tag{4}$$

$$\phi(-\infty) = 0, \qquad \phi(+\infty) = \kappa$$

E. Trofimchuk, V. Tkachenko and S. Trofimchuk,2008.




c
c*(h)
D*
h

Implicit equation of $c^*(h, g'(\kappa))$:

$$\frac{\beta - \alpha}{\beta e^{-\alpha ch} - \alpha e^{-\beta ch}} = \frac{q^2 + q}{q^2 + 1},$$

$\alpha < 0 < \beta$ roots of $z^2 - cz - 1 = 0$, $q := g'(\kappa)$.

Linearized equation about

- 0 equilibrium : $\phi''(t) - c\phi'(t) - \phi(t) + g'(0)\phi(t - ch) = 0$,
- κ equilibrium : $\phi''(t) - c\phi'(t) - \phi(t) + g'(\kappa)\phi(t - ch) = 0$,

Characteristic equations:

- 0 equilibrium: $\chi_0(z) := z^2 - cz - 1 + g'(0)e^{-chz} = 0$,
- κ equilibrium: $\chi_\kappa(z) := z^2 - cz - 1 + g'(\kappa)e^{-chz} = 0$,

Lemma

(a) *There exist* $c_0 = c_0(h, g'(0)) > 0$ *such that the characteristic equation* $\chi_0(z) = 0$ *has exactly two simple real roots* $0 < \lambda = \lambda(c) < \mu = \mu(c)$ *if and only if* $c > c_0(h)$. *Next, if* $c > c_0$, *then all complex roots* $\{\lambda_j\}_{j \geq 1}$ *of this equation are simple and can be ordered in such a way that*

$$\ldots \leq \Re(\lambda_3) \leq \Re(\lambda_4) \leq \Re(\lambda_2) = \Re(\lambda_1) < \lambda < \mu. \qquad (5)$$

Finally $c_0(h)$ *is a decreasing function, with* $c_0(+\infty) = 0$.

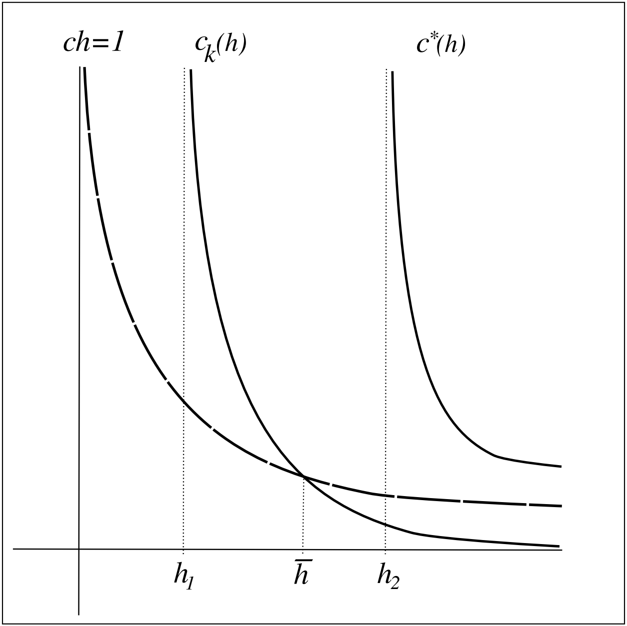
(b) *Let* $q := g'(\kappa)$. *There exist* $c_\kappa = c_\kappa(h) \in (0, +\infty]$ *such that the characteristic equation* $\chi_\kappa(z) = 0$ *has three real roots* $\lambda_1 \leq \lambda_2 < 0 < \lambda_3$ *if and only if* $c \leq c_\kappa(h)$. *Furthermore,* $c_\kappa(0) = +\infty$ *and* $c_\kappa(h)$ *is strictly decreasing in its domain, with* $c_\kappa(+\infty) = 0$.

Theorem

If $g(x) \leq g'(0)x$ *and* $g(x) \leq g'(\kappa)(x-k)+k$*, equation (4) has a monotone solution* $\phi(t)$ *if and only if*

$$(h,c) \in \mathcal{D}_{\mathcal{L}} := \{(h,c)/\ c_0(h) \leq c \leq c_\kappa(h)\}$$

Lemma



Implicit equations of $c_\kappa(h)$:

$$\frac{2+\sqrt{c^4h^2+4c^2h^2+4}}{c^2h^2|q|} = \exp\left(1+\frac{\sqrt{c^4h^2+4c^2h^2+4}-c^2h}{2}\right).$$

Theorem (1)

If g is not strictly increasing in $[0, \kappa]$, then equation 1 has a slowly oscillating traveling wave for each $(h, c) \in \mathcal{D}^* \backslash \mathcal{D}_{\mathcal{L}}$

Nicholson's Equation

$$u_t(t,x) = \Delta u(t,x) - \delta u(t,x) + pu(t-h,x)e^{-u(t-h,x)} \qquad (6)$$

- If $p/\delta > 1$ is a monostable reaction diffusion equation, with equilibria $u_0 = 0$ and $u_\kappa = ln(p/\delta)$

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- If $1 < p/\delta \leq e$ there exist a unique traveling wave solution and it must be monotone (showed using super- and sub-solution method for all $c > c_0(h)$ in [So-Zou, 2001]).
- if $e < p/\delta \leq e^2$ then there exist traveling waves solutions for al $(h,c) \in \mathcal{D}^*$ and there are monotone or slowly oscilating [E. Trofimchuk, V. Tkachenko and S. Trofimchuk, 2008], also there are monotone if $(h,c) \in \mathcal{D}_L \subset \mathcal{D}^*$ [A. Gomez, S. Trofimchuk, 2014] and slowly oscillating if $(h,c) \in \mathcal{D}^* \backslash \mathcal{D}_L$ (corollary of Theorem 1)

Theorem

If $p/\delta \in (e, e^2]$ *the region* $\mathcal{D}_{\mathcal{L}}$ *can have one of the following geometric forms with* $\nu_0 \approx 2.808..$ *and* h_a *defined by* $\delta h_a e^{\delta h_a} = [e \ln(p/e\delta)]^{-1}$

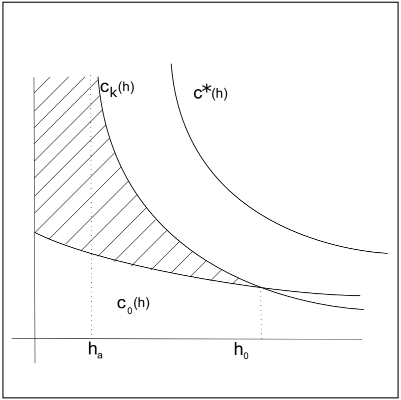

Figure: $p/\delta > \nu_0$

there exist a maximal delay h_0 to the monotonicity

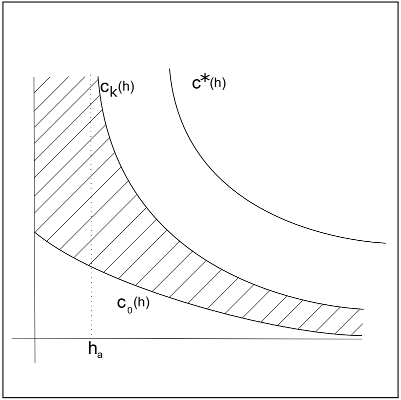

Figure: $p/\delta < \nu_0$

At minimum speed of propagation, the traveling waves solutions are monotone

Reference

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