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Reliable solution of problems with uncertain hysteresis operators

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- **In Engineering:** Control systems, Electronic circuits, Aerodynamics,. . .
- **In Mechanics:** Elastic hysteresis, Contact angle hysteresis, Bubble shape hysteresis, Adsorption hysteresis, Matric potential hysteresis, Magnetic hysteresis, Electrical hysteresis, Liquid-solid phase transitions,
- **In Biology:** Cell biology and genetics, Immunology, Neuroscience, Respiratory physiology, Voice and speech physiology, Ecology and epidemiology,. . .
- **In Economics:** Unemployment, Game theory, ...

Hysteresis operators

One-dimensional hysteresis operators on an interval $I = \langle 0, T\rangle$.
Functional dependence:

$$\mathcal{T}: \quad v \ \ (\text{function on } I) \quad \longmapsto \quad \mathcal{T}[v] \ \ (\text{function on } I)\,.$$

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Hysteresis operators are:

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- *locally monotone*:
  non-decreasing input causes non-decreasing output
  non-increasing input causes non-increasing output, i.e.
  $\mathcal{T}[v]'(t) \cdot v'(t) \geq 0$ for a. e. $t \in I$.

Basic stop and play hysteresis operators

Variational inequality definition

Let $r > 0$, $s_0 \in \langle -r, r\rangle$, $I = \langle 0, T\rangle$ and $u \in W^{1,1}(I)$

Let $s \in W^{1,1}(T)$ be the solution of the variational inequality

$$s(t) \in \langle -r, r\rangle \qquad s(0) = s_r^0$$

$$(s'(t) - u'(t))(\tilde{s} - s(t)) \geq 0 \quad \forall \tilde{s} \in \langle -r, r\rangle \quad t \in (0, T)$$

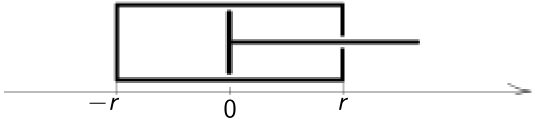
Then

$$\mathcal{S}_r[u](t) := s(t) \quad \text{is the stop operator}$$

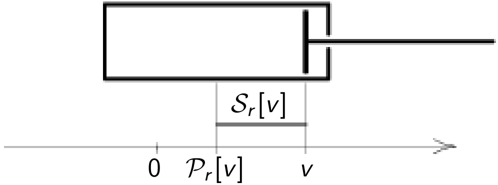
and the complement

$$\mathcal{P}_r[u](t) := u(t) - s(t) \quad \text{is the play operator}$$

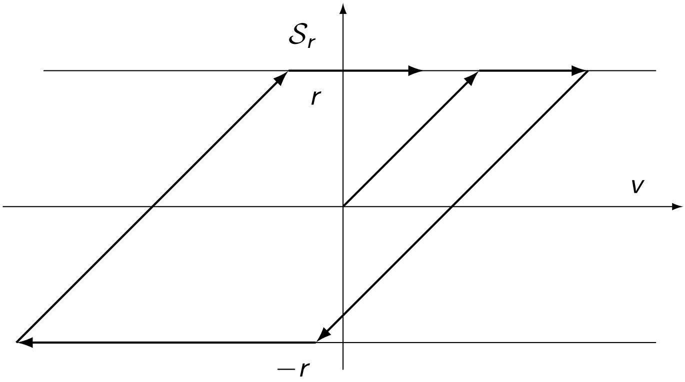
Graphic interpretation: Piston in cylinder model



Stop operator: position of the piston with respect to the cylinder.
Play operator: position of center of the cylinder

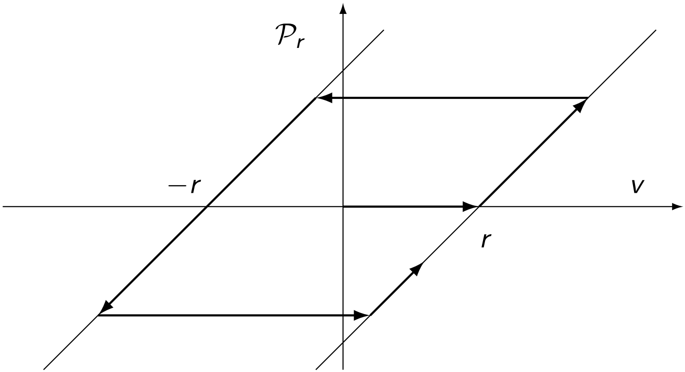


The stop operator



- concave increasing branches
- convex decreasing branches

The play operator



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Mechanical interpretation: elastic-friction model

The stop operator $\mathcal{S}_r[u](t) := s(t)$ serial composition

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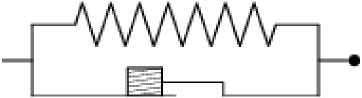
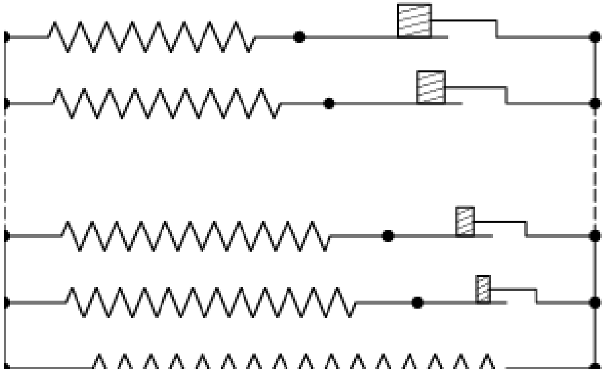


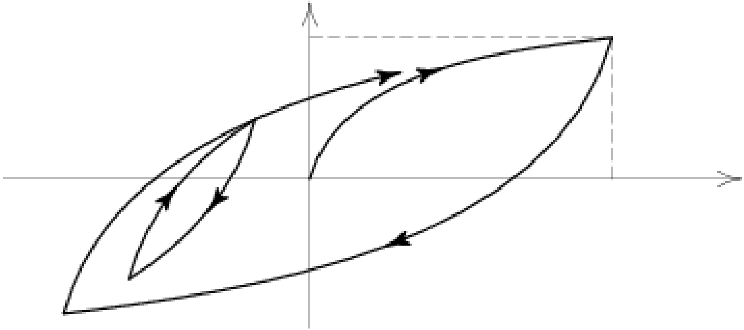
Paralel combination of stop operators



The Prandtl-Ishlinskii operator of stop type

$$\mathcal{F}[u] := \eta(0)u - \int_0^\infty \mathcal{S}_r[u]\mathrm{d}\eta(r)$$

$\eta(r)$ - positive, decreasing on $\langle 0, \infty)$



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The Prandtl-Ishlinskii operator of play type

$$\mathcal{G}[u] := \zeta(0)u + \int_0^\infty \mathcal{P}_r[u]\mathrm{d}\zeta(r)$$

$\zeta(r)$ - positive, increasing on $\langle 0, \infty)$, $\zeta(0) > 0$.

- convex increasing branches
- concave decreasing branches

Pair of mutually inverse operators

Let φ and ψ be mutually inverse functions on $\langle 0, \infty)$,
φ – concave, ψ – convex,

$$t = \varphi(s) \qquad \Longleftrightarrow \qquad s = \psi(t)$$

and $\eta(s) = \varphi'(s)$ non-increasing, on $\langle 0, \infty)$, $\eta(\infty) > 0$
and $\zeta(t) = \psi'(t)$ non-decreasing on $\langle 0, \infty)$, $\zeta(0) > 0$, we adopt

$$\beta \le \zeta(x, r) \le 1/\alpha.$$

The pair η, ζ is said to be in $PI(\alpha, beta)$.

The corresponding Prandtl-Ishlinskij operators are mutually inverse:

$$\sigma(t) = \mathcal{F}_\eta[e](t) \qquad \Longleftrightarrow \qquad e(t) = \mathcal{G}_\zeta[\sigma](t)$$

e – strain, deformation $\qquad$ σ – stress

Properties of hysteresis operator

The operators are first defined on piecewise monotone functions, by variational inequality can be extended to

$$\mathcal{F}_\eta, \mathcal{G}_\zeta \ : W^{1,\infty}(I) \to W^{1,\infty}(I)$$

and by continuity to

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The operators are Lipschitz continuous:

$$|\mathcal{F}_\eta[e_1](t) - \mathcal{F}_\eta[e_2](t)| \leq \left(\frac{1}{\beta} - \alpha\right) \|e_1 - e_2\|_{<0,t>}$$

$$|\mathcal{G}_\zeta[\sigma_1](t) - \mathcal{G}_\zeta[\sigma_2]|(t) \leq \left(\frac{1}{\alpha}\right) \|\sigma_1 - \sigma_2\|_{<0,t>}$$

Properties of hysteresis operator

The operators are local monotone:
Let $(\xi, \zeta) \in PI(\alpha, \beta)$ and $\sigma \in W^{1,1}(I)$.
Let $e := \mathcal{G}_\zeta[\sigma]$. Then for a. e. $t \in I$

$$\alpha \left(\frac{\mathrm{d}e}{\mathrm{d}t}(t) \right)^2 \leq \frac{\mathrm{d}e}{\mathrm{d}t}(t) \frac{\mathrm{d}\sigma}{\mathrm{d}t}(t) \leq \frac{1}{\beta} \left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}(t) \right)^2 ,$$

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and continuously dependent on distribution function ζ
Let $(\eta_i, \zeta_i) \in PI(\alpha, \beta)$ be two pairs of two distribution functions, $\mathcal{G}_{\zeta_1}, \mathcal{G}_{\zeta_2}$ the corresponding operators and $\sigma_1, \sigma_2 \in W^{1,1}(I)$ arbitrary input functions. Then

$$\|\mathcal{G}_{\zeta_1}[\sigma_1] - \mathcal{G}_{\zeta_2}[\sigma_2]\|_{[0,t]} \leq \zeta_1(\infty)\|\sigma_1 - \sigma_2\|_{[0,t]} + \int_0^{\|\sigma_2\|_{[0,t]}} |\zeta_1(r) - \zeta_2(r)| \mathrm{d}r .$$

Spatially dependent hysteresis operator

The Prandtl-Ishlinskii operators are determined by distribution functions $\xi(r)$, $\zeta(r)$.
In case of an inhomogeneous material the material properties depend even on space variable x. Thus both function η and ζ will depend in addition on space variable x, i.e.

$$\eta = \eta(x, r), \quad \zeta = \zeta(x, r).$$

Reliable solutions of problems with uncertain data

Boundary value problem modeling a real engineering problem contains data (constants, dimensions, functions, etc.) in constitutive relations, which are obtained by measurements and thus are not known exactly, the values are loaded with some errors.

The actual values are known to be in some extent.
This leads to the so-called problems with uncertain data.

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Solutions:
(a) stochastic approach
(b) deterministic approach by I. Babuška and I. Hlaváček

Ivan Hlaváček and Ivo Babuška 2007

The worst scenario method

We are looking for the worst scenario, the worst situation that can happen on the admissible uncertain data.

- $\mathcal{U}_{\mathrm{ad}}$ - the set of all admissible data – uncertain data
- u_a is the solution to the problem P_a with data a
- $\Phi(a, u)$ – functional evaluating dangerousness of the modelled situation.

Mathematic formulation of the problem:

Find $a^* \in \mathcal{U}_{\mathrm{ad}}$ and u_{a^*} solution of the problem P_a such that

$$\Phi(a^*, u_{a^*}) \geq \Phi(a, u_a) \quad u_a \text{ solution of } P_a \ \forall a \in \mathcal{U}_{\mathrm{ad}}.$$

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We aim to prove that such maximum exists.

Idea: choose a compact set $\mathcal{U}_{\mathrm{ad}}$ and the cost functional Φ continuously dependent on the data.

The cost functional

If the quantity u is continuous, the functional can be the value in a critical point x_0, e.g.

$$\Phi(a, u_a) = u_a(x_0) \qquad \text{where } u_a \text{ is solution of the problem } P_a$$

if u_a is in an L^p space only, we take e.g. its integral mean over a subset – critical area, e.g.

$$\Phi(a, u_a) = \frac{1}{|K|} \int_K u_a(x) \mathrm{d}x.$$

Problems with hysteresis operator

Scalar wave equation – vibration of an elastic bar

$$c\, u_{tt} = k \,\mathrm{div}\, \sigma + f \qquad \sigma = \mathcal{F}[u_x]$$

physical interpretation: vibration of an elasto-plastic bar or

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Diffusion equation

$$c\, u_t = k \operatorname{div} \sigma + f \qquad \sigma = \mathcal{F}[u_x]$$

with hysteresis operator

$$\mathcal{F} : e(t) \mapsto \sigma(t) = \mathcal{F}[e](t)$$

Diffusion problem

Initial boundary value problem

$$c\, u_t = \mathrm{div}\ \sigma + f \qquad x \in \Omega = (0,\ell)\,,\ \ t \in I = (0,T)$$

$$\sigma = \mathcal{F}[u_x] \quad \text{or} \quad u_x = \mathcal{G}[\sigma], \quad x \in \Omega\,,\ \ t \in I$$

completed by initial and boundary condition

$$u(x,0) = u_0(x), \quad x \in (0,\ell)\,, \qquad x \in \Omega\,,$$

$$u(0,t) = 0\,, \quad \sigma(\ell,t) = 0\,, \qquad t \in I = (0,T)\,.$$

Hypothesis:

- $c \in L^\infty((0,\ell))$ and $0 < c_m \le c(x) \le c_M$,
- $\eta, \zeta \in L^\infty(\Omega \times \langle 0,\infty))$ and $(\eta,\zeta)(x, cdot) \in PI(\alpha,\beta)$,
- $f \in W^{1,1}(I, L^2(\Omega)$
- $u_0 \in W^{2,2}(\Omega)$ and satisfies compatibility conditions.

Existence of the solution

Proposition

The problem admits unique solution $u(x,t), \sigma(x,t)$ satisfying

$$u, \sigma \in C(\Omega \times I)\,, \quad u_x \in L^2(\Omega; C(I))\,, u_t, \sigma_x \in L^\infty(I, L^2(\Omega))$$

and are bounded in the corresponding norms.

The set of admissible data

The constitutive parameters e.g. $c(x)$ are in a compact sets, i.e. closed bounded intervals, e.g.

$$c(x) = c_0 \in \langle c_{min}, c^{max} \rangle$$

or piecewise constant on given intervals.
Any sequence of the set contains a uniformly converging subsequence.

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or piecewise constant on given intervals.
Any sequence of the set contains a uniformly converging subsequence.

The distribution functions $\zeta(x, r)$ are piecewise constant in x and non-decreasing in r, constant outside a fixed interval $\langle 0, R \rangle$ and bounded: $\zeta(x, r) \in \langle \beta, 1/\alpha \rangle$.
Any sequence of the set contains a subsequence converging in L^∞.

Worst scenario method

Proposition

Under introduced assumptions the worst scenario has solution, i.e. the critical functional $\Phi(a, u)$ *attains its maximum value* a^*, u^**. on the set of admissible data* $\mathcal{U}_{\mathrm{ad}}$*.*

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Main steps of the proof:

- Take a sequence $\Phi(a_k, u_{a_k})$ converging to the supremum.
- Due to compactness of $\mathcal{U}_{\mathrm{ad}}$ a convergent subsequence is chosen.
- Due to continuity the subsequence tends to the maximum, which is finite.

Scalar wave equation

Problem of vibration of an elastoplastic bar $(0, \ell) = J$

$$\rho\, u_{tt} = \sigma_x + f \qquad \sigma = \mathcal{F}[u_x]$$

with boundary and initial conditions is reformulated into a first order system

$$\rho\, v_t = \sigma_x + f \qquad e_t = v_x \qquad e = \mathcal{G}_\zeta[\sigma]$$

where $v := u_t$ $\ e := u_x$ with boundary conditions e.g. $v(0, t) = 0$, $\sigma(\ell, t) = 0$ and initial condition $v(x, 0) = v_0(x)$ and $\sigma(x, 0) = \sigma_0$. Data of the problem: $\rho(x) \in L^\infty((0, \ell))$, pair $(\eta(x, r), \zeta(x, r))$ of adjoint functions each in $L^\infty((0, \ell))$, $f \in W^{1,1}(0, T; (0, \ell))$, initial conditions.

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Proposition

The problem admits unique solution $u(x,t), v(x,t), \sigma(x,t)$ satisfying

- $v, \sigma \in C(\langle 0, T\rangle \times \langle 0, \ell\rangle)$
- $e \in L^2((0,\ell), C(\langle 0, T\rangle$
- $e_t, v_t, \sigma_t, v_x, \sigma_x \in L^\infty((0, T); L^2(0,\ell))$.

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References

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On Monday you could see Castle Veveří visited by W. Churchhill in 1906, 1907, 1908