Applications of critical points results to existence and multiplicity of solutions for elliptic problems with variable exponent

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Differential Equations and Applications
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DiffEq[&]App

$$-\Delta_{p(x)}u = \lambda f(x,u)$$

$$\Delta_{p(x)}u := div(|\nabla u|^{p(x)-2}\nabla u)$$

- electrorheological fluids
  - Ružička (1999, 2000), Acerbi-Mingione (2002), Acerbi-Mingione-Seregin (2004)
- thermorheological fluids
  - Antontsev-Rodrigues (2006)
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$\Omega \subset \mathbb{R}^N$ open, bounded, $p \in C(\bar{\Omega})$

$$1 < p^- := \inf_{x \in \Omega} p(x) \le p(x) \le p^+ := \sup_{x \in \Omega} p(x) < +\infty$$

$$L^{p(x)}(\Omega) := \left\{ u : \Omega \to \mathbb{R} : u \text{ measurable}, \rho_p(u) := \int_\Omega |u(x)|^{p(x)} dx < +\infty \right\}$$

$$\|u\|_{L^{p(x)}(\Omega)} := \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \le 1 \right\}$$

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$$W^{1,p(x)}(\Omega) := \left\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \right\}$$

$$\|u\|_{W^{1,p(x)}(\Omega)} := \|u\|_{L^{p(x)}(\Omega)} + \||\nabla u|\|_{L^{p(x)}(\Omega)}$$

$$W_0^{1,p(x)}(\Omega) := \overline{C_0^\infty(\Omega)}^{W^{1,p(x)}(\Omega)}$$

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$L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable, reflexive and uniformly convex Banach spaces.

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there exists $c_0 > 0$ such that

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Bonanno-C. -Complex Var. Elliptic Equ.-(2012)

$$c_0 \le k_{p^-}(|\Omega|+1)$$

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$$k_{p^-} \le \frac{N^{-\frac{1}{p^-}}}{\sqrt{\pi}}\left[\Gamma\left(1+\frac{N}{2}\right)\right]^{\frac{1}{N}}\left(\frac{p^- - 1}{p^- - N}\right)^{1-\frac{1}{p^-}}|\Omega|^{\frac{1}{N}-\frac{1}{p^-}}$$

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DiffEq[&]App

$p^- > N$ estimate of constant for embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow C^0(\overline{\Omega})$ with respect $\|\cdot\|_a$

$$\|u\|_{W_0^{1,p(x)}} := \inf\left\{\sigma > 0 : \int_\Omega \left|\frac{\nabla u(x)}{\sigma}\right|^{p(x)} dx \leq 1\right\}$$

$\|\cdot\|$ is equivalent to $\|\cdot\|_a$

$$\|u\|_a = \inf\left\{\sigma > 0 : \int_\Omega \left(\left|\frac{\nabla u(x)}{\sigma}\right|^{p(x)} + a(x)\left|\frac{u(x)}{\sigma}\right|^{p(x)}\right) dx \leq 1\right\}$$

$$c_0^* \leq k_{p^-}(|\Omega| + 1)\frac{[a_-]_{1/p} + 1}{[a_-]_{1/p}}$$

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$1 < p^- \leq p^+ < +\infty$

Embedding's theorem

If $p \in C(\bar{\Omega})$ with $p(x) > 1$ for each $x \in \bar{\Omega}$ and $q \in C(\bar{\Omega})$ with

$$1 < q(x) < p^*(x) := \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } \ p(x) < N \\ \infty & \text{if } \ p(x) \geq N \end{cases}$$

for all $x \in \Omega$, then there exists a compact embedding

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$$

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$1 < p^- \leq p^+ < +\infty$, estimate of constant for embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^1(\Omega)$

$p^- < N$, Bonanno-C. -J.M.A.A.-(2014)

$$k_1 \leq c_{p^{-*}} |\Omega|^{\frac{p^{-*}-1}{p^{-*}}} (|\Omega| + 1)$$

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]App

$1 < p^- \leq p^+ < +\infty$,estimate of constant for embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$

$q \in C(\bar{\Omega})$, $p^- < N$ and $q^+ < p^{-*}$, Bonanno-C. -J.M.A.A.-(2014)

$$k_q \leq (|\Omega| + 1)^2 c_{p^{-*}} |\Omega|^{\frac{p^{-*} - q^+}{p^{-*} q^+}}.$$

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$1 < p^- \leq p^+ < +\infty$, estimate of constant for embedding $(W^{1,p(x)}(\Omega), \|\cdot\|_a) \hookrightarrow L^1(\Omega)$

Ω open and convex and $p^- \neq N$, Barletta-C. -E.J.D.E.-(2013)

$$\bar{k_1} \leq \tilde{k}_{p^-,1}(1+|\Omega|)(1+\|a\|_\infty)^{\frac{1}{p^-}} \frac{1+[a_-]_{\frac{1}{p}}}{[a_-]_{\frac{1}{p}}}$$

$$(W^{1,p(x)}, \|\cdot\|_a)(\Omega) \underbrace{\hookrightarrow}_{(|\Omega|+1)(1+\|a\|_\infty)^{1/p^-}\frac{[a_-]_{1/p}+1}{[a_-]_{1/p}}} (W^{1,p^-}, \|\cdot\|_a)(\Omega) \underbrace{\hookrightarrow}_{\tilde{k}_{p^-,1}} L^1(\Omega)$$

- $\tilde{k}_{p^-,1}$ is the embedding's constant

$$(W^{1,p^-}(\Omega), \|\cdot\|_a) \hookrightarrow L^1(\Omega)$$

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$$\bar{k_1} \leq \tilde{k}_{p^-,1}(1+|\Omega|)(1+\|a\|_\infty)^{\frac{1}{p^-}} \frac{1+[a_-]_{\frac{1}{p}}}{[a_-]_{\frac{1}{p}}}$$

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DiffEq[&]App

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  - Dirichlet problem
    - multiple solutions
    - infinitely many solutions
  - Neumann-type differential inclusion
    - multiple solutions
- $1 < p^- \leq p^+ < +\infty$
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- precise interval of parameters Λ
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DiffEq[&]App

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DiffEq[&]App

- there exist $x_0 \in \Omega$ and $\tau > 0$ such that

$$B(x_0, \underbrace{\sup_{x\in\Omega} \overbrace{\delta(x)}^{\sup\{\delta>0:B(x,\delta)\subseteq\Omega\}}}_{\tau}) \subseteq \Omega$$

- $\omega_\tau := \tau^N \frac{\pi^{\frac{N}{2}}}{\frac{N}{2}\Gamma(\frac{N}{2})}$,
- fixed $\alpha > 0$ and $h \in C(\bar{\Omega})$ we put

$$[\alpha]^h := \max\{\alpha^{h^-}, \alpha^{h^+}\} \quad [\alpha]_h := \min\{\alpha^{h^-}, \alpha^{h^+}\}$$

- $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function,

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DiffEq[&]App

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On the structure of the critical set of non-differentiable functions with a weak compactness condition,
Appl. Anal., **89** (2010), 1–10.

$$\Downarrow$$

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admits at least three weak solutions

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- $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function

$$|f(x,t)| \leq c(1 + |t|^{s-1}), \; s \in [1, p^-[$$

- $\operatorname{ess\,inf}_{x\in\Omega} F(x,t) \geq 0$ for each $t \in \mathbb{R}$
- there exist $r > 0$, $\delta > 0$ with $r < \frac{1}{p^+}\left[\frac{2\delta}{\tau}\right]_p \omega_\tau \left(1 - \frac{1}{2^N}\right)$:

$$\alpha_r := \int_\Omega \sup_{|\xi| \leq c_0 \gamma_r} F(x,\xi)\, dx < \frac{p^- \operatorname{ess\,inf}_{x\in\Omega} F(x,\delta)}{\left[\frac{2\delta}{\tau}\right]^p (2^N - 1)} := \beta_\delta$$

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for each $\lambda \in \Lambda_{r,\delta} :=]\frac{1}{\beta_\delta}, \frac{1}{\alpha_r}[$, the problem $(D_{\lambda,f})$ admits at least three weak solutions.

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- $\sigma(p^+, N) = \frac{1-\bar{\mu}^N}{\bar{\mu}^N(1-\bar{\mu})^{p^+}} = \inf_{\mu \in]0,1[} \frac{1-\mu^N}{\mu^N(1-\mu)^{p^+}}$
- $\sigma(p^-, N) = \frac{1-\bar{\mu}^N}{\bar{\mu}^N(1-\bar{\mu})^{p^-}} = \inf_{\mu \in]0,1[} \frac{1-\mu^N}{\mu^N(1-\mu)^{p^-}}$
- $I(\tau, \bar{\mu}) := \int_{B(x_0,\tau)\setminus B(x_0,\bar{\mu}\tau)} (\tau - |x - x_0|)^{p(x)} \, dx$
- $\beta_+ := \frac{\sigma(p^+,N)}{\tau^{p^+}} + \|a\|_\infty \left(1 + \frac{I(\tau,\bar{\mu})}{\omega_\tau \bar{\mu}^N(\tau(1-\bar{\mu}))^{p^+}}\right)$
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- $I(\tau, \bar{\mu}) := \int_{B(x_0,\tau)\setminus B(x_0,\bar{\mu}\tau)} (\tau - |x - x_0|)^{p(x)}\, dx$
- $\beta_+ := \frac{\sigma(p^+,N)}{\tau^{p^+}} + \|a\|_\infty \left(1 + \frac{I(\tau,\bar{\mu})}{\omega_\tau \bar{\mu}^N(\tau(1-\bar{\mu}))^{p^+}}\right)$
- $\beta_- := \frac{\sigma(p^-,N)}{\tau^{p^-}} + \|a\|_\infty \left(1 + \frac{I(\tau,\bar{\mu})}{\omega_\tau \bar{\mu}^N(\tau(1-\bar{\mu}))^{p^-}}\right)$

DiffEq[&]App

$p^- > N$

- $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function
- $A := \liminf_{\xi \to +\infty} \dfrac{\int_\Omega \max_{|t| \leq \xi} F(x,t)\, dx}{\xi^{p^-}}$, $B := \limsup_{\xi \to +\infty} \dfrac{\int_{B(x_0, \bar{\mu}\tau)} F(x,\xi)\, dx}{\xi^{p^+}}$
- c_0^* embedding's constant of $(W_0^{1,p(x)}(\Omega), \|\cdot\|_a) \hookrightarrow C^0(\overline{\Omega})$

Bonanno-C. - Complex Variables and Elliptic Equations - (2012)

(i) $\operatorname{ess\,inf}_{x \in \Omega} F(x,\xi) \geq 0$ for each $\xi \geq 0$

(ii) $A < \dfrac{p^-}{\beta_+ p^+ {c_0^*}^{p^-} \omega_\tau \bar{\mu}^N} B$

$\Longrightarrow$ for each $\lambda \in \Lambda := \left] \dfrac{\beta_+ \omega_\tau \bar{\mu}^N}{B p^-}, \dfrac{1}{p^+ {c_0^*}^{p^-} A} \right[$, the problem $(D_{\lambda,a,f})$ admits a sequence of weak solutions which is unbounded in $W_0^{1,p(x)}(\Omega)$

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$$\begin{cases} -\Delta_{p(x)}u + a(x)|u|^{p(x)-2}u \in \lambda\partial F(x,u) \text{ in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \end{cases} \qquad (\tilde{N}_\lambda)$$

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- $f(\cdot, \xi)$ measurable for each $\xi \in \mathbb{R}$;
- $f(x, \cdot)$ locally essentially bounded for each $x \in \Omega$;
- there exist $q \in C(\bar{\Omega})$, with $1 < q^- \leq q^+ < p^-$ and $c > 0$ such that

$$|f(x,\xi)| \leq c(1 + |\xi|^{q(x)-1})$$

  for each $(x, \xi) \in \Omega \times \mathbb{R}$.
- $\bar{c_0}$ embedding's constant of $(W^{1,p(x)}(\Omega), \|\cdot\|_a) \hookrightarrow C^0(\overline{\Omega})$
- there exist $r > 0$, $\xi_1 \in \mathbb{R}$ with $r < \dfrac{a^-}{p^+} |\Omega| \, [|\xi_1|]_p$ such that

$$\int_\Omega \sup_{|\xi| \leq \bar{c_0}[rp^+]^{1/p}} F(x,\xi)\, dx < \frac{rp^-}{|\Omega| \, a^+ [|\xi_1|]^p} \int_\Omega F(x, \xi_1)\, dx$$

DiffEq[&]App

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$$\Lambda := \left] \frac{p^-}{|\Omega|\, a^+ [|\xi_1|]^p \int_\Omega F(x,\xi_1)\,dx}, \frac{r}{\int_\Omega \sup_{|\xi| \le \bar{c_0} [rp^+]^{1/p}} F(x,\xi)\,dx} \right[$$

C. , Livrea - Discrete and Continuous Dynamical Systems - (2012)

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Appl. Anal., **89** (2010), 1–10.

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$$\begin{cases} -\Delta_{p(x)} u = \lambda f(x,u) \ \text{ in } \ \Omega \\ u = 0 \ \text{ on } \ \partial\Omega \end{cases} \qquad (D_{\lambda,f})$$

admits at least three weak solutions

$1 < p^- \leq p^+ < +\infty$

Bonanno-C. - J.M.A.A. - (2014)

- $|f(x,t)| \leq a_1 + a_2|t|^{q(x)-1}$, $q \in C(\bar{\Omega})$, $1 < q(x) < p^*(x)$
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- $F(x,t) \geq 0$, $\forall (x,t) \in \Omega \times \mathbb{R}^+$
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$$r < \frac{1}{p^+}\left[\frac{2\delta}{\tau}\right]_p \omega_\tau \left(1 - \frac{1}{2^N}\right) \qquad \alpha_r < \frac{p^- \inf_{x\in\Omega} F(x,\delta)}{\left[\frac{2\delta}{\tau}\right]^p (2^N - 1)}$$

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DiffEq[&]App

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DiffEq[&]App

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G. Bonanno and P. Candito,
J. Differential Equations **244** (2008), 3031–3059.

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$$\begin{cases} -\Delta_{p(x)} u = \lambda(f(x,u) + \mu g(x,u)) \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases} \qquad (D_{\lambda,\mu,f,g})$$

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DiffEq[&]App

$1 < p^- \leq p^+ < +\infty$

$$\mathcal{H} := \{h : \Omega \times \mathbb{R} \to \mathbb{R} \text{ locally bounded} : (m_1), (m_2), (m_3) \text{ hold } \}$$

(m_1) $h(\cdot, t)$ measurable for each $t \in \mathbb{R}$;

(m_2) there exists $\Omega_0 \subseteq \Omega$ with $m(\Omega_0) = 0$ such that the set

$$D_h := \bigcup_{x \in \Omega \setminus \Omega_0} \{t \in \mathbb{R} : h(x, \cdot) \text{ is discontinuous at } t\}$$

has measure zero.

(m_3) the functions

$$h^-(x, z) := \lim_{\delta \to 0^+} \operatorname{ess\,inf}_{|\xi - z| < \delta} h(x, \xi), \quad h^+(x, z) := \lim_{\delta \to 0^+} \operatorname{ess\,sup}_{|\xi - z| < \delta} h(x, \xi)$$

are superpositionally measurable i.e. $h^-(\cdot, u(\cdot))$ and $h^+(\cdot, u(\cdot))$ are measurable provided $u : \Omega \to \mathbb{R}$ is measurable too

DiffEq[&]App

$1 < p^- \leq p^+ < +\infty$

- $f,\ g \in \mathcal{H}$
- $q, \bar{q}, \gamma \in C(\bar{\Omega})$, $1 \leq q(x) < p^*(x)$, $1 \leq \bar{q}(x) \leq \bar{q}^+ < p^-$, $p^+ < \min\{\gamma^-, p^{*-}\}$:
  - $0 \leq f(x,t), g(x,t) \leq c_1(1 + |t|^{q(x)-1})$, $\forall (x,t) \in \Omega \times \mathbb{R}$;
  - $0 \leq f(x,t) \leq \bar{c}_1(1 + t^{\bar{q}(x)-1})$, $\forall x \in \Omega$ and $t \geq 0$;
  - $\limsup_{t \to 0^+} \sup_{x \in \Omega} \frac{F(x,t)}{t^{\gamma(x)}} < +\infty$;
- $\exists h > 0$: $\inf_{x \in \Omega} F(x,h) > 0$;
- $\forall \mu > 0$, for a. e. $x \in \Omega$ and $\forall t \in D_f \cup D_g$

$$(f + \mu g)^-(x,t) \leq 0 \leq (f + \mu g)^+(x,t) \Longrightarrow (f + \mu g)(x,t) = 0$$

DiffEq[&]App

$1 < p^- \leq p^+ < +\infty$

Bonanno-C. - Math. Nachr. - (2011)

For each $\lambda > \frac{2}{p^-}(2^N - 1)\frac{[\frac{2h}{\tau}]^p}{\inf_{x\in\Omega} F(x,h)}$, there exists $\delta > 0$ such that, for every $\mu \in [0, \delta]$, the problem $(D_{\lambda,\mu,f,g})$ admits at least three non negative weak solutions.

DiffEq[&]App

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Bonanno G.,
A critical point theorem via the Ekeland variational principle,
Nonlinear Analysis, **75** (2012), 2992–3007.

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Barletta-C. - Electronic Journal of Differential Equations - (2013)

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Bonanno G.,
Relations between the mountain pass theorem and local minima,
Adv.Nonlinear Anal., **1** (2012), 205–220.

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DiffEq[&]App

$1 < p^- \leq p^+ < +\infty$

Barletta-C. - Electronic Journal of Differential Equations - (2013)

- $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function
- there exist $a_1, a_2 \in]0, +\infty[$ e $q \in C(\bar{\Omega})$, $1 < q(x) < p^*(x)$ for each $x \in \bar{\Omega}$ such that $|f(x,t)| \leq a_1 + a_2|t|^{q(x)-1}$ for each $(x,t) \in \Omega \times \mathbb{R}$;
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Sturm-Liouville equations involving discontinuous nonlinearities,
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$$\begin{cases} -\Delta_{p(x)}u + a(x)|u|^{p(x)-2}u = \lambda f(x,u) \ \text{ in } \ \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 \ \text{ on } \partial\Omega \end{cases} \qquad (N_{\lambda,a})$$

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Barletta-C.- O'Regan - Nonlinear Analysis: Real World Applications - (2016)

Let $f \in \mathcal{H}$, satisfying

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(f_3) for each $\lambda > 0$, for a.e. $x \in \Omega$ and each $z \in D_f$ the condition $\lambda f^-(x,z) \leq a(x)|z|^{p(x)-2}z \leq \lambda f^+(x,z)$ implies $\lambda f(x,z) = a(x)|z|^{p(x)-2}z$,

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Bonanno G. and Chinnì A.,
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