Applications of critical points results to existence and multiplicity of solutions for elliptic problems with variable exponent

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$$\Delta_{p(x)}u := div(|\nabla u|^{p(x)-2}\nabla u)$$

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 - Ružička (1999, 2000), Acerbi-Mingione (2002), Acerbi-Mingione-Seregin (2004)
- thermorheological fluids
 - Antontsev-Rodrigues (2006)
- image restoration
 - Levine (2005), Aboulaich-Meskine-Souissi (2008)

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Research group on variable exponent Lebesgue and Sobolev spaces www.helsinki.fi/ pharjule/varsob/links.shtml

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DiffEq[&]An

$$\Omega \subset {\rm I\!R}^N$$
 open, bounded, $p \in C(\bar{\Omega})$

$$1 < p^- := \inf_{x \in \Omega} p(x) \le p(x) \le p^+ := \sup_{x \in \Omega} p(x) < +\infty$$

$$\begin{split} L^{p(x)}(\Omega) &:= \left\{ u: \Omega \to \mathbb{R} : u \text{ measurable}, \rho_p(u) := \int_{\Omega} |u(x)|^{p(x)} dx < +\infty \right\} \\ \|u\|_{L^{p(x)}(\Omega)} &:= \inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \le 1 \right\} \end{split}$$

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DiffEq[&]App

4/47

$$W^{1,p(x)}(\Omega) := \left\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \right\}$$

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 $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W^{1,p(x)}_0(\Omega)$ are separable, reflexive and uniformly convex Banach spaces.

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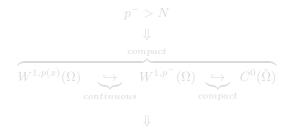
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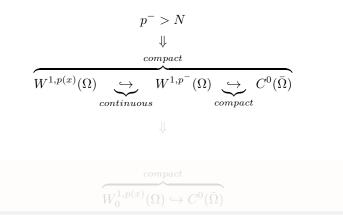
there exists $c_0 > 0$ such that

 $\|u\|_{C^0(\bar{\Omega})} \le c_0 \|u\|_{W_0^{1,p(x)}(\Omega)}$



6/47





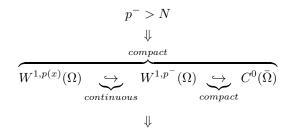
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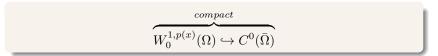
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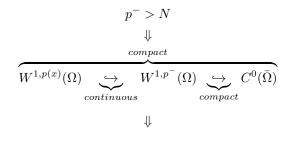
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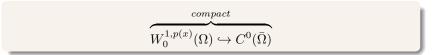
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$$\|u\|_{C^0(\bar{\Omega})} \leq c_0 \|u\|_{W^{1,p(x)}_0(\Omega)} \qquad \qquad \mathrm{DiffEq[\&]Approximation of the set of$$

 $p^- > N$ estimate of constant for embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow C^0(\overline{\Omega})$

Bonanno-C. -Complex Var. Elliptic Equ.-(2012)

 $c_0 \le k_{p^-}(|\Omega|+1)$

$$W_0^{1,p(x)}(\Omega) \underset{|\Omega|+1}{\hookrightarrow} W_0^{1,p^-}(\Omega) \underset{k_{p^-}}{\hookrightarrow} C^0(\bar{\Omega})$$
$$N^{-\frac{1}{p^-}} \left[\left(N \right) \right]^{\frac{1}{N}} \left(p^- - 1 \right)^{1-\frac{1}{p^-}}$$

$$r_{p^-} \leq \frac{m}{\sqrt{\pi}} \left[\Gamma\left(1 + \frac{m}{2}\right) \right] \quad \left(\frac{p^- - 1}{p^- - N}\right) \quad P \quad |\Omega|^{\frac{1}{N} - \frac{1}{p^-}}$$

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 $p^- > N$ estimate of constant for embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow C^0(\overline{\Omega})$ with respect $\|\cdot\|_a$

$$||u||_{W_0^{1,p(x)}} := \inf \left\{ \sigma > 0 : \int_{\Omega} \left| \frac{\nabla u(x)}{\sigma} \right|^{p(x)} dx \le 1 \right\}$$

 $\|\cdot\|$ is equivalent to $\|\cdot\|_a$

$$\|u\|_{a} = \inf\left\{\sigma > 0: \int_{\Omega} \left(\left|\frac{\nabla u(x)}{\sigma}\right|^{p(x)} + a(x)\left|\frac{u(x)}{\sigma}\right|^{p(x)}\right) dx \le 1\right\}$$

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$1 < p^- \le p^+ < +\infty$

Embedding's theorem

If $p \in C(\bar{\Omega})$ with p(x) > 1 for each $x \in \bar{\Omega}$ and $q \in C(\bar{\Omega})$ with

$$1 < q(x) < p^*(x) := \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N\\ \infty & \text{if } p(x) \ge N \end{cases}$$

for all $x \in \Omega$, then there exists a compact embedding

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$1 < p^- \leq p^+ < +\infty$, estimate of constant for embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^1(\Omega)$

$p^- < N$, Bonanno-C. -J.M.A.A.-(2014)

$$k_1 \le c_{p^{-*}} |\Omega|^{\frac{p^{-*}-1}{p^{-*}}} (|\Omega|+1)$$

$$W_0^{1,p(x)}(\Omega) \underset{|\Omega|+1}{\hookrightarrow} W_0^{1,p^-}(\Omega) \underset{c_{p^{-*}} \in |\Omega|}{\hookrightarrow} L^1(\Omega)$$

• $c_{p^{-*}}$ is the constant of the continuous embedding $W_0^{1,p^-}(\Omega) \hookrightarrow L^{p^{-*}}(\Omega)$

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]App

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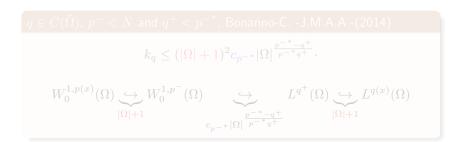
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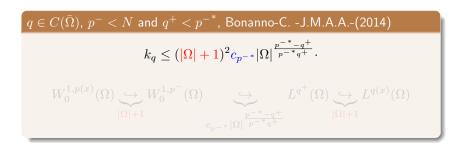
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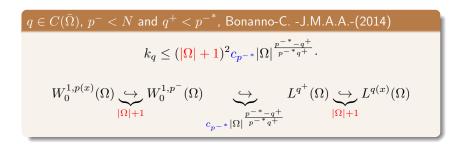


 $1 < p^{-} \leq p^{+} < +\infty$, estimate of constant for embedding $W_{0}^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$





 $1 < p^- \leq p^+ < +\infty$, estimate of constant for embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$





11/47

$1 < p^- \le p^+ < +\infty$, estimate of constant for embedding $(W^{1,p(x)}(\Omega), \|\cdot\|_a) \hookrightarrow L^1(\Omega)$

$$\begin{split} \Omega \text{ open and convex and } p^- \neq N, \text{ Barletta-C. -E.J.D.E.-(2013)} \\ \bar{k}_1 \leq \tilde{k}_{p^-,1}(1+|\Omega|)(1+\|a\|_{\infty})^{\frac{1}{p^-}} \frac{1+[a_-]_{\frac{1}{p}}}{[a_-]_{\frac{1}{p}}} \\ (W^{1,p(x)}, \|\cdot\|_a)(\Omega) & \longleftrightarrow \\ (|\Omega|+1)(1+\|a\|_{\infty})^{1/p^-} \frac{[a_-]_{1/p}+1}{[a_-]_{1/p}} \\ (W^{1,p^-}, \|\cdot\|_a)(\Omega) & \longleftrightarrow \\ \bar{k}_{p^-,1} \text{ is the embedding's constant} \\ (W^{1,p^-}(\Omega), \|\cdot\|_a) \hookrightarrow L^1(\Omega) \end{split}$$

$1 < p^- \le p^+ < +\infty$, estimate of constant for embedding $(W^{1,p(x)}(\Omega), \|\cdot\|_a) \hookrightarrow L^1(\Omega)$

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]App

 $1 < p^- \le p^+ < +\infty$, estimate of constant for embedding $(W^{1,p(x)}(\Omega), \|\cdot\|_a) \hookrightarrow L^1(\Omega)$

$$\begin{split} &\Omega \text{ open and convex and } p^- \neq N, \text{ Barletta-C. -E.J.D.E.-(2013)} \\ &\bar{k}_1 \leq \tilde{k}_{p^-,1} (1+|\Omega|) (1+\|a\|_{\infty})^{\frac{1}{p^-}} \frac{1+[a_-]_{\frac{1}{p}}}{[a_-]_{\frac{1}{p}}} \\ &(W^{1,p(x)}, \|\cdot\|_a)(\Omega) \underbrace{\hookrightarrow}_{(|\Omega|+1)(1+\|a\|_{\infty})^{1/p^-} \frac{[a_-]_{1/p}+1}{[a_-]_{1/p}}} (W^{1,p^-}, \|\cdot\|_a)(\Omega) \underbrace{\leftrightarrow}_{\tilde{k}_{p^-,1}} L^1(\Omega) \\ &\bullet \tilde{k}_{p^-,1} \text{ is the embedding's constant} \\ &(W^{1,p^-}(\Omega), \|\cdot\|_a) \hookrightarrow L^1(\Omega) \end{split}$$

 $q\in C(\bar{\Omega})$ and $q^+ < p^{-*}$, Ω open and convex and $p^-
eq N$, Barletta-C. -E.J.D.E.-(2013)

$$\bar{k_q} \le \tilde{k}_{p^-,q^+} (1+|\Omega|)^2 (1+||a||_{\infty})^{\frac{1}{p^-}} \frac{1+[a_-]_{\frac{1}{p}}}{[a_-]_{\frac{1}{p}}}$$

$$W^{1,p(x)}(\Omega) \underbrace{\hookrightarrow}_{(|\Omega|+1)(1+\|a\|_{\infty})^{1/p^{-}}\frac{[a_{-}]_{1/p}+1}{[a_{-}]_{1/p}}} W^{1,p^{-}}(\Omega) \underbrace{\hookrightarrow}_{\tilde{k}_{p^{-},q^{+}}} L^{q^{+}}(\Omega) \underbrace{\hookrightarrow}_{|\Omega|+1} L^{q(x)}(\Omega)$$

• \tilde{k}_{p^-,q^+} is the constant for embedding

$$(W^{1,p^{-}}(\Omega), \|\cdot\|_a) \hookrightarrow L^{q^{+}}(\Omega)$$

ADD

 $q \in C(\overline{\Omega})$ and $q^+ < p^{-*}$, Ω open and convex and $p^- \neq N$, Barletta-C. -E.J.D.E.-(2013) $\bar{k_q} \le \tilde{k}_{p^-,q^+} (1+|\Omega|)^2 (1+||a||_{\infty})^{\frac{1}{p^-}} \frac{1+|a_-|_{\frac{1}{p}}}{[a_-]_1}$ $\underbrace{\hookrightarrow}_{(|\Omega|+1)(1+\|a\|_{\infty})^{1/p^{-}}\frac{[a_{-}]_{1/p}+1}{[a_{-}]_{1/p}}}W^{1,p^{-}}(\Omega)\underbrace{\hookrightarrow}_{\tilde{k}_{p^{-},q^{+}}}L^{q^{+}}(\Omega)\underbrace{\hookrightarrow}_{|\Omega|+1}L^{q(s)}(\Omega)$.]App

 $q \in C(\overline{\Omega})$ and $q^+ < p^{-*}$, Ω open and convex and $p^- \neq N$, Barletta-C. -E.J.D.E.-(2013) $\bar{k_q} \le \tilde{k}_{p^-,q^+} (1+|\Omega|)^2 (1+||a||_{\infty})^{\frac{1}{p^-}} \frac{1+|a_-|_{\frac{1}{p}}}{|a_-|_1}$ $W^{1,p^{-}}(\Omega) \underset{\tilde{k}_{p^{-},q^{+}}}{\hookrightarrow} L^{q^{+}}(\Omega) \underset{|\Omega|+1}{\hookrightarrow} L^{q(x)}(\Omega)$ $W^{1,p(x)}(\Omega)$ $(|\Omega|+1)(1+||a||_{\infty})^{1/p} - \frac{[a_{-}]_{1/p}+1}{[a_{-}]_{1/p}}$.]App

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The problems

• $p^- > N$

- Dirichlet problem
 - multiple solutions
 - infinitely many solutions
- Neumann-type differential inclusion
 - multiple solutions
- $1 < p^- \le p^+ < +\infty$
 - Dirichlet problem
 - multiple solutions
 - Neumann problem
 - multiple solutions

precise interval of parameters Λ



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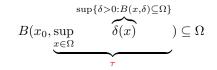
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• there exist $x_0 \in \Omega$ and $\tau > 0$ such that



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$$\omega_{\tau} := \tau^N \frac{\pi^{\frac{\lambda^2}{2}}}{\frac{N}{2}\Gamma(\frac{N}{2})},$$

• fixed $\alpha > 0$ and $h \in C(\bar{\Omega})$ we put

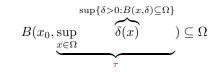
$$[\alpha]^h := \max\{\alpha^{h^-}, \alpha^{h^+}\} \quad [\alpha]_h := \min\{\alpha^{h^-}, \alpha^{h^+}\}$$

• $h: \Omega imes \mathbb{R} o \mathbb{R}$ is a Carathéodory function,

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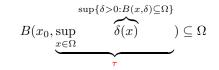
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15/47

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ΔT

$$B(x_0, \sup_{\substack{x \in \Omega \\ x \in \Omega}} \overbrace{\delta(x)}^{\sup\{\delta > 0: B(x, \delta) \subseteq \Omega\}}) \subseteq \Omega$$

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$$\omega_{\tau} := \tau^{N} \frac{\pi^{\frac{\Lambda^{2}}{2}}}{\frac{N}{2}\Gamma(\frac{N}{2})},$$

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Bonanno G. and Marano S. A.

On the structure of the critical set of non-differentiable functions with a weak compactness condition, Appl. Anal., **89** (2010), 1–10.

$$\downarrow
\begin{cases}
-\Delta_{p(x)}u = \lambda f(x, u) \text{ in } \Omega \\
u = 0 \text{ on } \partial\Omega
\end{cases}$$
(D_{\lambda,f})

admits at least three weak solutions



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• $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function $|f(x,t)|\leq c(1+|t|^{s-1}),\ s\in[1,p^-[$

• ess inf
$$_{x \in \Omega} F(x,t) \ge 0$$
 for each $t \in \mathbb{R}$

• there exist r > 0, $\delta > 0$ with $r < \frac{1}{p^+} \left[\frac{2\delta}{\tau}\right]_p \omega_{\tau} \left(1 - \frac{1}{2^N}\right)$:

$$\alpha_r := \int_{\Omega} \sup_{|\xi| \le c_0 \gamma_r} F(x,\xi) \ dx < \frac{p^- \operatorname{ess\,inf}_{-x \in \Omega} F(x,\delta)}{\left[\frac{2\delta}{\tau}\right]^p \left(2^N - 1\right)} := \beta_{\delta}$$





• $f: \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function

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Bonanno, C.- Le Matematiche (2011)

for each $\lambda \in \Lambda_{r,\delta} :=]\frac{1}{\beta_{\delta}}, \frac{1}{\alpha_r}[$, the problem $(D_{\lambda,f})$ admits at least three weak solutions.

c₀ is the embedding's constant of W^{1,p(x)}₀(Ω) → C⁰(Ω̄)
 γ_r := [p⁺r]^{1/p}





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• c_0 is the embedding's constant of $W_0^{1,p(x)}(\Omega) \hookrightarrow C^0(\overline{\Omega})$ • $\gamma_r := [p^+r]^{\frac{1}{p}}$





$$\begin{cases} -\Delta_{p(x)}u = \lambda f(x, u) \text{ in } \Omega\\ u = 0 \text{ su } \partial\Omega \end{cases}$$
 $(D_{\lambda, f})$

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$$(D_{\lambda,a,f})$$

• $a \in L^{\infty}(\Omega)$ with ess $\inf_{\Omega} a \ge 0$



19/47



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Antonia Chinnì (University of Messina) Applications of critical points results to existence and DiffEqApp

19/47



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19/47

$p^- > N$

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20/47

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$p^- > N$

•
$$\sigma(p^+, N) = \frac{1 - \bar{\mu}^N}{\bar{\mu}^N (1 - \bar{\mu})^{p^+}} = \inf_{\mu \in]0,1[} \frac{1 - \mu^N}{\mu^N (1 - \mu)^{p^+}}$$

• $\sigma(p^-, N) = \frac{1 - \bar{\mu}^N}{\bar{\mu}^N (1 - \bar{\mu})^{p^-}} = \inf_{\mu \in]0,1[} \frac{1 - \mu^N}{\mu^N (1 - \mu)^{p^-}}$
• $I(\tau, \bar{\mu}) := \int_{B(x_0, \tau) \setminus B(x_0, \bar{\mu}\tau)} (\tau - |x - x_0|)^{p(x)} dx$
• $\beta_+ := \frac{\sigma(p^+, N)}{\tau^{p^+}} + ||a||_{\infty} \left(1 + \frac{I(\tau, \bar{\mu})}{\omega_\tau \bar{\mu}^N (\tau (1 - \bar{\mu}))^{p^+}}\right)$
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 $p^- > N$

• $f: \Omega \times \mathbb{R} \to \mathbb{R}$ is an L^1 -Carathéodory function

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$$A := \liminf_{\xi \to +\infty} \frac{\int_{\Omega} \max_{|t| \le \xi} F(x,t) \, dx}{\xi^{p^-}}, \ B := \limsup_{\xi \to +\infty} \frac{\int_{B(x_0,\bar{\mu}\tau)} F(x,\xi) \, dx}{\xi^{p^+}}$$

• c_0^* embedding's constant of $(W_0^{1,p(x)}(\Omega), \|\cdot\|_a) \hookrightarrow C^0(\overline{\Omega})$

Bonanno-C. - Complex Variables and Elliptic Equations - (2012)

]App

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•
$$c_0^* \text{ embedding's constant of } (W_0^{1, p(x)}(\Omega), \|\cdot\|_a) \hookrightarrow C^0(\overline{\Omega})$$

Bonanno-C. - Complex Variables and Elliptic Equations - (2012)

$$\begin{array}{l} (i) \ \mbox{ess inf}_{x\in\Omega}F(x,\xi) \geq 0 \ \mbox{for each } \xi \geq 0 \\ (ii) \ \ A < \frac{p^-}{\beta_+ p^+ c_0^* p^- \omega_\tau \bar{\mu}^N} B \\ \Longrightarrow \ \mbox{for each } \lambda \in \Lambda := \left] \frac{\beta_+ \omega_\tau \bar{\mu}^N}{Bp^-}, \frac{1}{p^+ c_0^* p^- A} \right[, \ \mbox{the problem } (D_{\lambda,a,f}) \\ \ \ \mbox{admits a sequence of weak solutions which is unbounded in} \\ W_0^{1,p(x)}(\Omega) \end{array}$$

1App

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(i) ess
$$\inf_{x \in \Omega} F(x, \xi) \ge 0$$
 for each $\xi \ge 0$
(ii) $A < \frac{p^-}{\beta_+ p^+ c_0^* p^- \omega_\tau \overline{\mu}^N} B$
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1App

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• c_0^* embedding's constant of $(W_0^{1,p(x)}(\Omega), \|\cdot\|_a) \hookrightarrow C^0(\overline{\Omega})$

Bonanno-C. - Complex Variables and Elliptic Equations -(2012)

1App

$$p^- > N$$

• $f: \Omega \times \mathbb{R} \to \mathbb{R}$ is an L^1 -Carathéodory function

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Bonanno-C. - Complex Variables and Elliptic Equations -(2012)

(i) ess
$$\inf_{x\in\Omega} F(x,\xi) \ge 0$$
 for each $\xi \ge 0$
(iii) $A^* < \frac{p^-}{\beta_- p^+ c_0^{*p^+} \omega_\tau \bar{\nu}^N} B^*$.
 \implies for each $\lambda \in \Lambda^* := \left] \frac{\beta_- \omega_\tau \bar{\nu}^N}{B^* p^-}, \frac{1}{p^+ c_0^{*p^+} A^*} \right[$, the problem $(D_{\lambda,a,f})$
admits a sequence of distinct weak solutions which strongly
converges to zero in $W_0^{1,p(x)}(\Omega)$.

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Bonanno G. and Candito P.

J. Differential Equations 244 (2008), 3031–3059.

Bonanno G. and Marano S. A., Appl. Anal., **89** (2010), 1–10.

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$$\begin{cases} -\Delta_{p(x)}u + a(x)|u|^{p(x)-2}u \in \lambda \partial F(x,u) \text{ in } \Omega\\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega \end{cases}$$
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- $f(\cdot,\xi)$ measurable for each $\xi \in \mathbb{R}$;
- $f(x, \cdot)$ locally essentially bounded for each $x \in \Omega$;
- $\bullet\,$ there exist $q\in C(\bar{\Omega}),$ with $1< q^-\leq q^+< p^-$ and c>0 such that

 $|f(x,\xi)| \le c(1+|\xi|^{q(x)-1})$

for each $(x,\xi) \in \Omega \times \mathbb{R}$.

- $\bar{c_0}$ embedding's constant of $(W^{1,p(x)}(\Omega), \|\cdot\|_a) \hookrightarrow C^0(\overline{\Omega})$
- there exist r>0, $\xi_1\in {\rm I\!R}$ with $r< \displaystyle rac{a^-}{p^+}\,|\Omega|\,[|\xi_1|]_p$ such that

$$\int_{\Omega} \sup_{|\xi| \le \bar{c_0} [rp^+]^{1/p}} F(x,\xi) \, dx < \frac{rp^-}{|\Omega| \, a^+ [|\xi_1|]^p} \int_{\Omega} F(x,\xi_1) \, dx$$



$$\Lambda := \left] \frac{p^-}{|\Omega| \, a^+[|\xi_1|]^p \int_{\Omega} F(x,\xi_1) \, dx}, \frac{r}{\int_{\Omega} \sup_{|\xi| \le \tilde{c_0}[rp^+]^{1/p}} F(x,\xi) \, dx} \right|$$

C., Livrea - Discrete and Continuous Dynamical Systems - (2012)

for each $\lambda \in \Lambda$, the problem (\tilde{N}_{λ}) admits at least three distinct solutions.





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A critical point theorem via the Ekeland variational principle, Nonlinear Analysis, **75** (2012), 2992–3007.

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Antonia Chinnì (University of Messina) Applications of critical points results to existence and DiffEqApp

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- $f:\Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function
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Relations between the mountain pass theorem and local minima, Adv.Nonlinear Anal., **1** (2012), 205–220.

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- $F(x,t) \le c(1+|t|^{\gamma(x)}), \ \gamma \in C(\bar{\Omega}), \ 1 < \gamma^{-} \le \gamma^{+} < p^{-}$
- $F(x,t) \ge 0, \ \forall (x,t) \in \Omega \times \mathbb{R}^+$
- there exist r > 0 and $\delta > 0$ such that

$$r < \frac{1}{p^+} \left[\frac{2\delta}{\tau}\right]_p \omega_\tau \left(1 - \frac{1}{2^N}\right) \quad \alpha_r < \frac{p^- \inf_{x \in \Omega} F(x, \delta)}{\left[\frac{2\delta}{\tau}\right]^p (2^N - 1)}$$

for each $\lambda \in \Lambda_{\tau, \delta} := \left[\frac{\left[\frac{2\delta}{\tau}\right]^p (2^N - 1)}{\left[\frac{2\delta}{\tau}\right]^p (2^N - 1)} - \frac{1}{2^N}\right]$ the problem (D)

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$$\Rightarrow \text{ for each } \lambda \in \Lambda_{r,\delta} := \left| \frac{\left|\frac{2\delta}{\tau}\right|^p (2^N - 1)}{p^{-} \inf_{x \in \Omega} F(x,\delta)}, \frac{1}{\alpha_r} \right|, \text{ the problem } (D_{\lambda,f})$$

admits three weak solutions.

$$\alpha_r := \frac{1}{r} \left\{ a_1 k_1 (p^+)^{\frac{1}{p^-}} [r]^{\frac{1}{p}} + \frac{a_2}{q^-} [k_q]^q (p^+)^{\frac{q^+}{p^-}} \left[[r]^{\frac{1}{p}} \right]^q \right\}$$

Existence of a non trivial weak solution Existence of two distinct weak solutions Existence of three weak solutions Multiple solutions with discontinuous non linear term

$1 < p^- \le p^+ < +\infty$

Bonanno-C. - J.M.A.A. - (2014)

- $|f(x,t)| \le a_1 + a_2 |t|^{q(x)-1}, \ q \in C(\overline{\Omega}), \ 1 < q(x) < p^*(x)$
- $F(x,t) \leq c(1+|t|^{\gamma(x)}), \ \gamma \in C(\bar{\Omega}), \ 1 < \gamma^- \leq \gamma^+ < p^-$
- $F(x,t) \ge 0, \ \forall (x,t) \in \Omega \times \mathbb{R}^+$
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$$\Rightarrow$$
 for each $\lambda \in \Lambda_{r,\delta} := \left| \frac{\left|\frac{2\pi}{p}\right|^{r} \left(2^{(r-1)}\right)}{p^{-} \inf_{x \in \Omega} F(x,\delta)}, \frac{1}{\alpha_r} \right|$, the problem $(D_{\lambda,f})$

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G. Bonanno and P. Candito,

J. Differential Equations **244** (2008), 3031–3059.

\Downarrow

$$\begin{cases} -\Delta_{p(x)}u = \lambda(f(x,u) + \mu g(x,u)) \text{ in } \Omega\\ u = 0 \text{ on } \partial\Omega \end{cases}$$
 $(D_{\lambda,\mu,f,g})$

admits at least three weak solutions

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 $\mathcal{H} := \{h : \Omega \times \mathbb{R} \to \mathbb{R} \text{ locally bounded} : (m_1), (m_2), (m_3) \text{ hold } \}$

 $\begin{array}{ll} (m_1) & h(\cdot,t) \mbox{ measurable for each } t \in \mathbb{R}; \\ (m_2) & \mbox{there exists } \Omega_0 \subseteq \Omega \mbox{ with } m(\Omega_0) = 0 \mbox{ such that the set} \end{array}$

$$D_h := \bigcup_{x \in \Omega \setminus \Omega_0} \{ t \in \mathbb{R} : h(x, \cdot) \text{ is discontinuous at } t \}$$

has measure zero.

(m_3) the functions

$$h^{-}(x,z) := \lim_{\delta \to 0^{+}} \mathrm{ess} \inf_{|\xi - z| < \delta} h(x,\xi), \ h^{+}(x,z) := \lim_{\delta \to 0^{+}} \mathrm{ess} \sup_{|\xi - z| < \delta} h(x,\xi)$$

are superpositionally measurable i.e. $h^-(\cdot, u(\cdot))$ and $h^+(\cdot, u(\cdot))$ are measurable provided $u: \Omega \to \mathbb{R}$ is measurable too DiffEq[81App

Existence of a non trivial weak solution Existence of two distinct weak solutions Existence of three weak solutions Multiple solutions with discontinuous non linear term

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$$(f + \mu g)^{-}(x,t) \le 0 \le (f + \mu g)^{+}(x,t) \Longrightarrow (f + \mu g)(x,t) = 0$$

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Bonanno-C. - Math. Nachr. - (2011)

For each $\lambda > \frac{2}{p^-}(2^N-1)\frac{[\frac{2h}{\tau}]^p}{\inf_{x\in\Omega}F(x,h)}$, there exists $\delta > 0$ such that, for every $\mu \in [0,\delta]$, the problem $(D_{\lambda,\mu,f,g})$ admits at least three non negative weak solutions.



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Bonanno G.,

A critical point theorem via the Ekeland variational principle, Nonlinear Analysis, **75** (2012), 2992–3007.

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Barletta-C. - Electronic Journal of Differential Equations - (2013)

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Existence of a non trivial weak solution Existence of two distinct weak solutions Existence of a non trivial weak solution with discontinuous non linear term

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Bonanno G.,

Relations between the mountain pass theorem and local minima, Adv.Nonlinear Anal., **1** (2012), 205–220.

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DiffEa[&]Ap

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DiffEa[&]A

Existence of a non trivial weak solution Existence of two distinct weak solutions Existence of a non trivial weak solution with discontinuous non linear term

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Bonanno G., D'Aguì G. and Winkert P. Sturm-Liouville equations involving discontinuous nonlinearities. Minimax Theory and its Applications, **01**, 1 (2015).

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Existence of a non trivial weak solution Existence of two distinct weak solutions Existence of a non trivial weak solution with discontinuous non linear term

$1 < p^- \le p^+ < +\infty$

Bonanno G., D'Aguì G. and Winkert P. Sturm-Liouville equations involving discontinuous nonlinearities, Minimax Theory and its Applications, **01**, 1 (2015).

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$$\begin{cases} -\Delta_{p(x)}u + a(x)|u|^{p(x)-2}u = \lambda f(x,u) \text{ in } \Omega\\\\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \end{cases}$$

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Multiple solutions for Neumann problem

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Barletta-C.- O'Regan - Nonlinear Analysis: Real World Applications - (2016)

Let $f \in \mathcal{H}$, satisfying

(f₂) there exist $a_1, a_2 \in [0, +\infty[$ and $q \in C(\overline{\Omega})$ with $1 < q(x) < p^*(x)$ for each $x \in \overline{\Omega}$, such that

$$|f(x,t)| \le a_1 + a_2 |t|^{q(x)-1}$$

(f₃) for each
$$\lambda > 0$$
, for a.e. $x \in \Omega$ and each $z \in D_f$ the condition $\lambda f^-(x,z) \le a(x)|z|^{p(x)-2}z \le \lambda f^+(x,z)$ implies $\lambda f(x,z) = a(x)|z|^{p(x)-2}z$,

$$(f_{\epsilon})$$

$$\limsup_{t \to 0^+} \frac{\int_{\Omega} F(x,t) \, dx}{t^{p^-}} = +\infty.$$

⇒ there exists $\lambda^* > 0$ such that for every $\lambda \in]0, \lambda^*[$ the problem $(N_{\lambda,a})$ admits at least one non trivial weak solution.

DittEq[&]App

$1 < p^- \le p^+ < +\infty$

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DittEq[&]App

Existence of a non trivial weak solution Existence of two distinct weak solutions Existence of a non trivial weak solution with discontinuous non linear term

$1 < p^- \le p^+ < +\infty$

$$\lambda^* = \frac{1}{a_1 \bar{k_1}(p^+)^{\frac{1}{p^-}} + \frac{a_2}{q^-} \left[\bar{k_q}\right]^q (p^+)^{\frac{q^+}{p^-}}}$$





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