On effective asymptotic formulas for two-dimensional wave equation with localized right-hand side.

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Statement of the problem

Consider the Cauchy problems for non-homogeneous wave equation with variable coefficients:

$$\frac{\partial^2 \eta}{\partial t^2} - \nabla c^2(x) \nabla \eta = Q(t,x) \quad \eta|_{t=0} = 0, \quad \eta_t|_{t=0} = 0, \qquad t \geq 0, \quad x \in \mathbb{R}^2 \tag{0.1}$$

with two right-hand sides:

1. **The instant source**

$$Q(t,x) = \delta'(t) V\left(\frac{x}{\mu}\right). \tag{0.2}$$

2. **The source localized in time**

$$Q(t,x) = \lambda^2 g_0'(\lambda t) V\left(\frac{x}{\mu}\right). \tag{0.3}$$

Here $c(x) > 0$ is smooth, $V(y)$ is a smooth function in $\mathbb{R}^2$ such that $|V^{(\alpha)}(y)| \leq C_\alpha(1 + |y|)^{-|\alpha|-\kappa}$ for some $\kappa > 1$, and $g_0(\tau)$, $\tau \in [0, \infty)$ is such that

$$g_0(0) = 0, \quad \int_0^\infty g_0(\tau)\, d\tau = 1, \quad |g_0^{(k)}(\tau)| \leq C_k e^{-\nu\tau} \qquad \text{for some } \nu > 0$$

Meaning of parameters

Applications: tsunami waves appearing due to local bottom displacements.

Usually the size of the source (of order 10 to 100 km) is much smaller than the size of the basin (or ocean) (of order 1000 km). Their ratio, μ can be considered as a small parameter. This allows us to use asymptotic methods.

The Cauchy problem for the instant source $Q(t,x) = \delta'(t)V\left(\frac{x}{\mu}\right)$ is also called 'piston' model. It acts instantly, because by the Duhamel principle the solution η can be found from

$$\frac{\partial^2 \eta}{\partial t^2} - \nabla c^2(x)\nabla\eta = 0, \quad \eta|_{t=0} = V\left(\frac{x}{\mu}\right) \quad \eta_t|_{t=0} = 0, \quad x \in \mathbb{R}^2. \tag{0.4}$$

Parameter λ is large $\lambda \gg 1$, and $1/\lambda$ characterizes the duration of the source. The case $Q(t,x) = \lambda^2 g_0(\lambda t)V(x/\mu)$ corresponds to a 'slightly smeared in time' source. It becomes instant source as $\lambda \to \infty$.

Model examples

As a model example we can take various examples of $g_0(\tau)$, for instance

$$g_0(\tau) = e^{-\tau}\mathcal{P}(\tau), \quad \mathcal{P}(\tau) = \sum_{k=1}^{n} \frac{\mathcal{P}_k}{k!}\tau^k, \quad \mathcal{P}_0 = 0, \quad \sum_{k=1}^{n} \mathcal{P}_k = 1. \tag{0.5}$$

To obtain asymptotic formulas as explicit as possible, we take the following 'simple source':

$$V(y) = A\left(1 + \left(\frac{y_1}{b_1}\right)^2 + \left(\frac{y_2}{b_2}\right)^2\right)^{-3/2}. \tag{0.6}$$

(After S. Dotsenko, B. Sergievskiy, L. Cherkasov, S. Sekerzh-Zenkovich).
Its Fourier transform is:

$$\widetilde{V}(p) = Ab_1b_2e^{-\sqrt{b_1^2p_1^2+b_2^2p_2^2}}.$$

Instant source: short-wave asymptotics

Let us describe the asymptotics of the solution

$$\frac{\partial^2\eta}{\partial t^2} - \nabla c^2(x)\nabla\eta = 0, \quad \eta|_{t=0} = V\left(\frac{x}{\mu}\right) \quad \eta_t|_{t=0} = 0, \quad x \in \mathbb{R}^2.$$

We follow the work by Dobrokhotov, Minenkov, Nazaikinskii and Tirozzi (2012).

Consider the Hamiltonian system in the phase space $\mathbb{R}^4$:

$$\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x}, \qquad H(x,p) = c(x)\sqrt{p_1^2 + p_2^2}.$$

Let $P(\psi,t), X(\psi,t)$ be its solutions with initial conditions

$$p\big|_{t=0} = \mathbf{n}(\psi), \quad x\big|_{t=0} = 0, \qquad \mathbf{n}(\psi) = (\cos\psi, \sin\psi), \qquad \psi \in [0, 2\pi)$$

Define a smooth curve $\Gamma_t \subset \mathbb{R}^4$ by the equations $x = X(\psi,t), p = P(\psi,t)$. Its projection $\gamma_t \subset \mathbb{R}^2_x$ onto the configuration space called the *wave front* may have singularities (focal points, where $X_\psi = 0$ and self-intersections).

The phase function

Assume that γ_t has no focal points and no self intersections. Define the function $\psi = \psi(x,t)$ by the condition that the vector $x - X(\psi,t)$ is orthogonal to the wave front at point $X(\psi,t)$:

$$\langle x - X(\psi(x,t),t), X_\psi(\psi(x,t),t)\rangle = 0.$$

Let us set

$$S(x,t) = \langle P(\psi(x,t),t), x - X(\psi(x,t),t)\rangle.$$

Let us also introduce the Morse index $m(\psi_0,t)$ of the trajectory $X(\psi_0,t)$ as the number of zeros of $|X_\psi(\psi_0,\tau)|$ on the interval $\tau \in [0,t]$.

Theorem 1 *In a neighborhood of the front γ_t but outside a neighborhood of focal points the solution of the problem*

$$\frac{\partial^2\eta}{\partial t^2} - \nabla c^2(x)\nabla\eta = 0, \quad \eta|_{t=0} = V\left(\frac{x}{\mu}\right) \quad \eta_t|_{t=0} = 0, \quad x \in \mathbb{R}^2,$$

where

$$V(y) = A\left(1 + \left(\frac{y_1}{b_1}\right)^2 + \left(\frac{y_2}{b_2}\right)^2\right)^{-3/2}$$

up to lower order terms as $\mu \to 0$ reads

$$\eta(t) = \sqrt{\mu}\mathrm{Re}\left[\frac{e^{-i\pi m(\psi_0,t)/2}}{\sqrt{|X_\psi(t,\psi)|}}\frac{\sqrt{c(0)}}{\sqrt{c(X(t,\psi))}}F_{\mathrm{ins}}\left(\frac{S(x,t)}{\mu},\psi\right)\right]\Bigg|_{\psi=\psi(t,x)}.$$

Here

$$F_{\mathrm{ins}}(z,\psi) = \frac{Aib_1b_2}{2\sqrt{2}(z+i\beta(\psi))^{3/2}},$$

where

$$\beta(\psi) = \sqrt{b_1^2\cos^2\psi + b_2^2\sin^2\psi}.$$

Source localized in time: propagating and transient part of solution

Consider the problem

$$\frac{\partial^2 \eta}{\partial t^2} - \nabla c^2(x)\nabla \eta = g'(t)V(x/\mu) \quad \eta|_{t=0} = 0, \quad \eta_t|_{t=0} = 0, \qquad t \geq 0, \quad x \in \mathbb{R}^2$$

(where $g(t) = \lambda g_0(\lambda t)$). By the Duhamel principle the solution can be written in a form

$$\eta = \int_0^t w(t, \xi, x)\, d\xi,$$

where $w(t, \xi, x)$ is the solution of the problem

$$\frac{\partial^2 w}{\partial t^2} - \nabla c^2(x)\nabla w = 0, \quad w|_{t=\xi} = g(\xi)V\left(\frac{x}{\mu}\right) \quad w_t|_{t=\xi} = 0, \quad x \in \mathbb{R}^2.$$

The operator $-\nabla c^2(x)\nabla$ is self-adjoint and non-negative in $L^2(\mathbb{R}^2)$. Let us define $D = \sqrt{-\nabla c^2(x)\nabla}$. Then the equation takes the form: $w_{tt} + D^2 w = 0$. This is an ordinary equation with operator coefficients.

The solution of the equation $w_{tt} + D^2 w = 0$ can be written out in terms of one-parameter group of operators e^{iDt} in $L^2(\mathbb{R}^2)$:

$$w = \mathrm{Re}\Big[e^{iD(t-\xi)} g(\xi)\Big] V\left(\frac{x}{\mu}\right), \qquad \eta = \mathrm{Re}\Big[\int_0^t e^{iD(t-\xi)} g(\xi)\, d\,\xi\Big] V\left(\frac{x}{\mu}\right).$$

Now we re-write the solution as follows

$$\eta = \mathrm{Re}\Big[e^{iDt} \int_0^\infty e^{-iD\tau} g(\tau)\, d\,\tau\Big] V\left(\frac{x}{\mu}\right) - \mathrm{Re}\Big[\int_0^\infty e^{-iD\xi} g(t+\xi)\, d\,\xi\Big] V\left(\frac{x}{\mu}\right).$$

After introducing the function

$$G(\zeta, t) = \int_0^\infty e^{-i\zeta\xi} g(t+\xi)\, d\,\xi = e^{i\zeta t} \int_t^\infty e^{-i\zeta\xi} g(\xi)\, d\,\xi,$$

we finally come at

$$\eta = \sqrt{2\pi}\, \mathrm{Re}\big(e^{iDt} \tilde{g}(D)\big) V\left(\frac{x}{\mu}\right) - \mathrm{Re}(G(D,t)) V\left(\frac{x}{\mu}\right).$$

Here $\tilde{g}(\zeta)$ is the Fourier transform of $g(t)$.

In the sum

$$\eta = \sqrt{2\pi}\,\mathrm{Re}\big(e^{iDt}\tilde{g}(D)\big)V\left(\frac{x}{\mu}\right) - \mathrm{Re}(G(D,t))V\left(\frac{x}{\mu}\right).$$

the first term is the propagating part, and the second one is the transient part. We are interested only in the former one:

$$\eta_{\mathrm{pr}} = \sqrt{2\pi}\,\mathrm{Re}\big(e^{iDt}\tilde{g}(D)\big)V\left(\frac{x}{\mu}\right) \tag{0.7}$$

We shall write η instead of η_{pr}. Note that $\eta(t)$ is the solution of the problem:

$$\eta_{tt} + D^2\eta = 0, \quad \eta\big|_{t=0} = \sqrt{2\pi}\mathrm{Re}\,\tilde{g}(D)V, \quad \eta_t\big|_{t=0} = -\sqrt{2\pi}\mathrm{Im}\,\tilde{g}(D)DV.$$

This initial data is called *the equivalent source.*

So the problem is reduced to a form suitable for asymptotic analysis.

Theorem 2 *In a neighborhood of the front γ_t but outside a neighborhood of focal points the solution of the problem*

$$\frac{\partial^2\eta}{\partial t^2} - \nabla c^2(x)\nabla\eta = g'(t)V(x/\mu), \quad \eta|_{t=0} = 0 \quad \eta_t|_{t=0} = 0, \quad x \in \mathbb{R}^2,$$

where

$$g(t) = \lambda g_0(\lambda t), \quad g_0(\tau) = e^{-\tau}\sum_{k=1}^{n}\frac{\mathcal{P}_k}{k!}\tau^k, \quad \mathcal{P}_0 = 0, \quad \sum_{k=1}^{n}\mathcal{P}_k = 1$$

up to lower order terms as $\mu \to 0$ reads

$$\eta(t) = \sqrt{\mu}\mathrm{Re}\Big[\frac{e^{-i\pi m(\psi_0,t)/2}}{\sqrt{|X_\psi(t,\psi)|}}\frac{\sqrt{c(0)}}{\sqrt{c(X(t,\psi))}}F_{\mathrm{loc}}\Big(\frac{S(x,t)}{\mu},\psi\Big)\Big]\Big|_{\psi=\psi(t,x)}.$$

Here

$$F_{\mathrm{loc}}(z,\psi) = -\frac{Ab_1b_2\Lambda^{3/2}\sqrt{\pi}i}{\sqrt{2}}\sum_{k=1}^{n}\frac{\mathcal{P}_k w^{k-1/2}}{k!}(-\frac{\partial}{\partial w})^k[\sqrt{w}e^w\big(1-\mathrm{erf}(\sqrt{w})\big)]$$

where

$$\Lambda = \frac{\lambda\mu}{c(0)}, \quad w = \Lambda(z+i\beta), \quad \beta(\psi) = \sqrt{b_1^2\cos^2\psi + b_2^2\sin^2\psi}, \quad \mathrm{erf}\, u = \frac{2}{\sqrt{\pi}}\int_0^u e^{-x^2}\,dx.$$

Some remarks

1. The only difference between Theorems 1 and 2 is the profile functions F_{ins} and F_{loc}. Let us compare them

$$F_{\text{ins}}(z,\psi) = \frac{Aib_1b_2}{2\sqrt{2}(z+i\beta(\psi))^{3/2}}$$

and

$$F_{\text{loc}}(z,\psi) = -\frac{Ab_1b_2\Lambda^{3/2}\sqrt{\pi}i}{\sqrt{2}}\sum_{k=1}^{n}\frac{\mathcal{P}_k w^{k-1/2}}{k!}(-\frac{\partial}{\partial w})^k[\sqrt{w}e^w\big(1-\text{erf}(\sqrt{w})\big)]$$

Obviously the source localized in time tends to an instant source as $\Lambda \to \infty$. It is an easy exercise to see that F_{loc} equals F_{loc} modulo lower order terms as $\Lambda \to \infty$.

2. The formula for F_{loc} is much more cumbersome than that for F_{ins}. It is also less pleasant for numerical computations. Our idea is to construct a simple approximation for F_{loc} starting from F_{ins} with help operator tricks described above.

Calculating F_{loc} when $\tilde{g}$ is a polynomial

Let us go back to the propagating part of the solution

$$\eta_{\text{loc}} = \sqrt{2\pi}\,\text{Re}\big(e^{iDt}\tilde{g}(D)\big)V\left(\frac{x}{\mu}\right).$$

Note that $\tilde{g} = 1$ corresponds to the case of instant source: η_{ins}. It can be shown that if $\tilde{g}$ is a polynomial, then using the identity: $e^{iDt}(iD)^k = \left(\frac{\partial}{\partial t}\right)^k e^{iDt}$ we can express η_{loc} via η_{ins}.

After some calculations we obtain the connection between two profile functions

Lemma 1 *Assume that* $\tilde{g}(\zeta) = \sum_{k=0}^{N} C_k (i\zeta)^k$. *Then*

$$F_{\text{loc}} = \sum_{k=0}^{N} \overline{C}_k \frac{(-1)^k}{\Lambda^k} \frac{\partial^k F_{\text{ins}}}{\partial z^k}.$$

Approximation with Laguerre polynomials

Now the question arises: how to approximate $\tilde{g}$ with polynomials $\tilde{g}_N$ in an optimal way? Let us take $t = 0$, then

$$\eta = \sqrt{2\pi}\mathrm{Re}\,\tilde{g}(D)V\big(\frac{x}{\mu}\big), \quad \eta_t = \sqrt{2\pi}\mathrm{Re}\,\big(iD\tilde{g}(D)\big)V\big(\frac{x}{\mu}\big),$$

which can be re-written as follows:

$$\eta = \frac{\mu^2}{\sqrt{2\pi}}\mathrm{Re}\int \tilde{g}(c_0|p|)e^{-|p|\mu\beta(\psi)}e^{ipx}\,dp = \frac{1}{\sqrt{2\pi}}\mathrm{Re}\int \tilde{g}_0\Big(\frac{|p|}{\Lambda}\Big)e^{-|p|\beta(\psi)}e^{\frac{ipx}{\mu}}\,dp$$

$$\eta_t = \frac{1}{\sqrt{2\pi}}\mathrm{Re}\int ic_0|p|\tilde{g}_0\Big(\frac{|p|}{\Lambda}\Big)e^{-|p|\beta(\psi)}e^{\frac{ipx}{\mu}}\,dp.$$

Thus η and η_t are real parts of Fourier transform of the functions

$$\frac{1}{\sqrt{2\pi}}\tilde{g}_0\big(\frac{\rho}{\Lambda}\big)e^{-\rho\beta(\psi)}, \quad \frac{ic(0)}{\sqrt{2\pi}}\rho\tilde{g}_0\Big(\frac{\rho}{\Lambda}\Big)e^{-\rho\beta(\psi)},$$

where $\rho = |p|$. If we replace $\tilde{g}_0$ with Laguerre polynomials, we get the best approximation for these expressions on $\rho \in [0, \infty)$.

Consider the Laguerre polynomials $L_j(p)$:

$$L_j(\rho) = \sum_{l=0}^{j} \binom{j}{l} \frac{(-1)^l \rho^l}{l!}, \quad L_j^K(\rho) = \sqrt{K} L_j(K\rho) \qquad (K > 0)$$

Let $\tilde{g}_N$ be approximation of $\tilde{g}_0$ with a linear combination of Laguerre polynomials:

$$\tilde{g}_N\big(\frac{|p|}{\Lambda}\big) = \sum_{j=0}^{N} C_j L_j^\beta(|p|), \quad C_j = \int_0^\infty \tilde{g}_0\big(\frac{\rho}{\Lambda}\big) L_j^\beta(\rho) e^{-\beta\rho}\, d\rho$$

After some calculations we obtain the final formula for the profile function

$$F_{\text{loc}}(z, \psi) = \frac{A i b_1 b_2}{2\sqrt{2}} \sum_{l=0}^{N} \overline{B_l} (-i)^l \frac{\beta^{l+1/2}(2l+1)!!}{2^l (z + i\beta)^{3/2+l}}, \quad B_l = \frac{1}{l!} \sum_{j=l}^{N} \binom{j}{l} C_j$$

Numerical calculations

We have taken:

$$V(y) = \left(1 + \left(\frac{y_1}{2}\right)^2 + \left(\frac{y_2}{3}\right)^2\right)^{-3/2}$$

and

$$g_0(\tau) = e^{-\tau}\mathcal{P}(\tau), \quad \mathcal{P}(\tau) = 2\tau - \frac{\tau^2}{2},$$

and compared the asymptotic formula

$$F_{\mathrm{loc}}(z,\psi) = -\frac{Ab_1b_2\Lambda^{3/2}\sqrt{\pi}i}{\sqrt{2}}\sum_{k=1}^{n}\frac{\mathcal{P}_k w^{k-1/2}}{k!}(-\frac{\partial}{\partial w})^k[\sqrt{w}e^w\big(1-\mathrm{erf}(\sqrt{w})\big)]$$

with a formula obtained by Laguerre approximation

$$F_{\mathrm{loc}}(z,\psi) = \frac{Aib_1b_2}{2\sqrt{2}}\sum_{l=0}^{N}\overline{B_l}(-i)^l\frac{\beta^{l+1/2}(2l+1)!!}{2^l(z+i\beta)^{3/2+l}}, \quad B_l = \frac{1}{l!}\sum_{j=l}^{N}\binom{j}{l}C_j.$$